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REGULARIZATION BY NONLOCAL CONDITIONS OF THE INCORRECT PROBLEMS FOR DIFFERENTIAL-OPERATOR EQUATIONS OF THE FIRST ORDER

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1. SECTION 1

We consider the differential-operator equation

$$\frac{du(t)}{dt} + Au(t) = 0 \tag{1}$$

in the interval $(0, T)$, where u is the function of a variable $t \in [0, T]$ with values in the Hilbertian space H, and A is a non-bounded self-adjoint operator in H.

It is assumed that the operator A does not have a fixed sign and has a bounded reciprocal operator A^{-1} within H. Without any loss in generality, for simplicity we assume that the operator A has a point spectrum. An example of this type of operator is the operator in $L_2(\Omega)$ generated by the operator Δ^2 (Δ is the Laplacian given in a bounded dimain $\Omega \subset R^2$) and boundary conditions at the boundary $\partial\Omega$, giving together with the operator Δ^2 a self-adjoint noncoercitive problem (see [1], p.397). The Cauchy problem for the eq. (1) with conditions of the form

$$u(0) = \chi \quad \text{or} \quad u(T) = \chi \tag{2}$$

is not correct in sense of Hadamar - Petrovski.

In the present work we have shown that the conditions (2) may be understood as follows. We introduce nonlocal boundary conditions to eq. (1) in the following way :

$$\alpha u(0) + (1-\alpha)u(T) = \chi, \quad \chi \in H, \quad 0 < \alpha < 1. \tag{3}$$

Then, the problem (1), (3) has a strongly generalized solution $U(t,\alpha)$ having the following properties :

$$\lim_{\alpha \to 0} ||u(T,\alpha) - \chi|| = 0; \qquad \lim_{\alpha \to 1} ||u(0,\alpha) - \chi|| = 0, \tag{4}$$

where $|| \cdot ||$ is norm in space H.

2. SECTION 2

Now we demonstrate that the problem (1), (3) has a strongly generalized solution.

Let $\{\lambda_n\}_{n\geq 1}$ be positive eigenvalues of the operator A, and $\{\mu_n\}_{n\geq 1}$ be negative eigenvalues of that operator. Let us denote the eigenvectors of the operator A corresponding to the eigenvalues $\{\lambda_n\}_{n\geq 1}$ by $\{v_n\}_{n\ \geq 1}$ and those corresponding to the eigenvalues $\{\mu_n\}_{n\geq 1}$ - $\{w_n\}_{n\geq 1}$.

Eigenvectors v_n and w_n of the operator A are forming in H a complete orthogonal system. Without loss in generality, we assume that $||v_n|| = 1$ and $||w_n|| = 1$.

Let the vector $\chi \in H$ be represented as

$$\chi = \sum_{n=1}^{\infty} c_n v_n + \sum_{n=1}^{\infty} d_n w_n, \tag{5}$$

where

$$c_n = (v_n, \chi), \qquad d_n = (w_n, \chi), \tag{6}$$

and $(.,.)$ is a scolar product in H.

The solution $u(t,\alpha)$ of eq. (1) is soughtin the form

$$u(t,\alpha) = \sum_{n=1}^{\infty} \varphi_n e^{-\lambda_n t} v_n + \sum_{n=1}^{\infty} \psi_n e^{\mu_n (T-t)} w_n. \tag{7}$$

substituting this expression into the conditions of (3) and using the representation of (5), we obtain the following equality :

$$\sum_{n=1}^{\infty} [\alpha + (1-\alpha)e^{-\lambda_n T}]\varphi_n v_n + \sum_{n=1}^{\infty} [\alpha e^{\mu_n T} + (1-\alpha)]\psi_n w_n =$$

$$\sum_{n=1}^{\infty} c_n v_n + \sum_{n=1}^{\infty} d_n w_n. \tag{8}$$

From this it is inferred that

$$\varphi_n = [\alpha + (1-\alpha)e^{-\lambda_n T}]^{-1} c_n, \tag{9}$$

$$\psi_n = [\alpha e^{\mu_n T} + (1-\alpha)]^{-1} d_n. \tag{10}$$

Substituting the values φ_n and ψ_n into (7), we have

$$\begin{aligned} u(t,\alpha) &= \sum_{n=1}^{\infty} e^{-\lambda_n t}[\alpha + (1-\alpha)e^{-\lambda_n T}]^{-1}(v_n,\chi)v_n \\ &+ \sum_{n=1}^{\infty} e^{\mu_n (T-t)}[\alpha e^{\mu_n T} + (1-\alpha)]^{-1}(w_n,\chi)w_n. \end{aligned} \tag{11}$$

Since

$$\left| e^{-\lambda_n t}\left[\alpha + (1-\alpha)e^{-\lambda_n T}\right]^{-1}\right| \leq \frac{1}{\alpha} \tag{12}$$

$$-\left| e^{\mu_n (T-t)}\left[\alpha e^{\mu_n T} + (1-\alpha)\right]^{-1}\right| \leq \frac{1}{1-\alpha}, \tag{13}$$

then

$$\begin{aligned} &\sup_{0\le t\le T}\left\| \sum_{n=m}^{m+p} e^{-\lambda_n t}[\alpha + (1-\alpha)e^{-\lambda_n T}]^{-1}(\nu_n,\chi)\nu_n \right\|^2 \\ &+ \sup_{0\le t\le T}\left\| \sum_{n=m}^{m+p} e^{\mu_n (T-t)}[\alpha e^{\mu_n T} + (1-\alpha)]^{-1}(w_n,\chi)w_n \right\|^2 \\ &\leq \frac{1}{\alpha^2}\sum_{n=m}^{m+p}(v_n,\chi)^2 + \frac{1}{(1-\alpha)^2}\sum_{n=m}^{m+p}(w_n,\chi)^2 \end{aligned} \tag{14}$$

and consequently the series (11) converge to the space norm $C([0,T],H)$ for the functions continuous on $[0,T]$ and possessing the values in H.

Thus, for any $\chi \in H$ the problem (1), (3) has a solution strongly generalixed in $C([0,T],H)$, that is representable as a series (11), and for this solution the following estimate is correct:

$$\sup_{0\le t\le T}\|u(t,\alpha)\|^2 \le \frac{1}{\alpha^2}\sum_{n=1}^{\infty}(\nu_n,\chi)^2+\frac{1}{(1-\alpha)^2}\sum_{n=1}^{\infty}(w_n,\chi)^2.$$

3. SECTION 3

In this paragraph it is demonstrated that the solution $u(t,\alpha)$ posesses the properties of (4). Firstly, it is shown that

$$\lim_{\alpha\to 0}||\chi-u(T,\alpha)||=0, \qquad \forall\chi\in H. \tag{15}$$

Let us consider the difference

$$\begin{aligned}\chi-u(T,\alpha)= \quad & \sum_{n=1}^{\infty}\frac{\alpha+(1-\alpha)e^{-\lambda_nT}-e^{-\lambda_nT}}{\alpha+(1-\alpha)e^{-\lambda_nT}}(v_n,\chi)v_n \\ & +\sum_{n=1}^{\infty}\frac{\alpha e^{\mu_nT}+(1-\alpha)-1}{\alpha e^{\mu_nT}+(1-\alpha)}(w_n,\chi)w_n.\end{aligned} \tag{16}$$

For all $\alpha\in[0,1]$ the value

$$\frac{\alpha+(1-\alpha)e^{-\lambda_nT}-e^{-\lambda_nT}}{\alpha+(1-\alpha)e^{-\lambda_nT}}=\frac{\alpha(1-e^{-\lambda_nT})}{\alpha+(1-\alpha)e^{-\lambda_nT}} \tag{17}$$

is nonnegative and nondecreasing with α since its derivative with respect to α is as follows:

$$\frac{(1-e^{-\lambda_nT})e^{-\lambda_nT}}{(\alpha+(1-\alpha)e^{-\lambda_nT})^2}\ge 0.$$

Therefore, the value (17) takes the greatest value at $\alpha=1$, i.e.

$$0\le\frac{\alpha(1-e^{-\lambda_nT})}{\alpha+(1-\alpha)e^{-\lambda_nT}}\le 1-e^{-\lambda_nT}\le 1. \tag{18}$$

For all $\alpha\in[0,1/2]$ the value

$$\frac{\alpha e^{\mu_nT}+(1-\alpha)-1}{\alpha e^{\mu_nT}+(1-\alpha)}=\frac{\alpha(e^{\mu_nT}-1)}{\alpha e^{\mu_nT}+(1-\alpha)} \tag{19}$$

is nonpositive and nonincreasing with α inasmuch as its derivative with respect to α is as follows :

$$\frac{(e^{\mu_nT}-1)}{(\alpha e^{\mu_nT}+(1-\alpha))^2}\le 0.$$

Hence the value of (19) on the segment $[0, 1/2]$ is the lowest when $\alpha = 1/2$, i.e.

$$-1 \leq \frac{e^{\mu_n T} - 1}{e^{\mu_n T} + 1} \leq \frac{\alpha e^{\mu_n T} + (1-\alpha) - 1}{\alpha + (1-\alpha)e^{-\lambda_n T}} \leq 0. \tag{20}$$

Based on the inequalities (18) and (20), from eq. (16) it is inferred that uniformly in $\alpha \in [0, 1/2]$ the following estimate is true :

$$\begin{aligned} \|\chi - u(t,\alpha)\|^2 &= \sum_{n=1}^{\infty} \frac{(\alpha + (1-\alpha)e^{-\lambda_n T} - e^{-\lambda_n T})^2}{(\alpha + (1-\alpha)e^{-\lambda_n T})^2} (v_n, \chi)^2 \\ &\quad + \sum_{n=1}^{\infty} \frac{(\alpha e^{\mu_n T} + (1-\alpha) - 1)^2}{(\alpha e^{\mu_n T} + (1-\alpha))^2} (w_n, \chi)^2 \\ &\leq \sum_{n=1}^{\infty} (v_n, \chi)^2 + \sum_{n=1}^{\infty} (w_n, \chi)^2 = \|\chi\|^2. \end{aligned} \tag{21}$$

Now we demonstrate that

$$\lim_{\alpha \to 0} \|\chi - u(T,\alpha)\|^2 = 0, \quad \forall \chi \in M. \tag{22}$$

where M is some set dense in H.

Then by virtue of the Banach-Steinhaus theorem, (15) is derived from (21) and (22).

As a set M we take all χ of the form [see also (5)] :

$$\chi = \sum_{n=1}^{N} c_n v_n + \sum_{n=1}^{\infty} d_n w_n, \quad \forall N < \infty.$$

Thereupon,

$$\begin{aligned} \|\chi - u(t,\alpha)\|^2 &= \sum_{n=1}^{N} \frac{\alpha^2 (1 - e^{-\lambda_n T})^2}{(\alpha + (1-\alpha)e^{-\lambda_n T})^2} (v_n, \chi)^2 \\ &\quad + \sum_{n=1}^{\infty} \frac{\alpha^2 (e^{\mu_n T} - 1)^2}{(\alpha e^{\mu_n T} + (1-\alpha))^2} (w_n, \chi)^2 \\ &\leq \frac{\alpha^2 e^{2\lambda_N T}}{(1-\alpha)^2} \sum_{n=1}^{N} (v_n, \chi)^2 + \frac{\alpha^2}{(1-\alpha)^2} \sum_{n=1}^{\infty} (w_n, \chi)^2 \\ &= \frac{\alpha^2 e^{2\lambda_N T}}{(1-\alpha)^2} \|\chi\|^2 \end{aligned}$$

and for such χ (22) is valid.

Now we prove that

$$\lim_{\alpha\to 1} \|u(0,\alpha)-\chi\| = 0, \qquad \forall \chi \in H. \tag{23}$$

Let us consider the difference

$$\chi - u(0,\alpha) = \sum_{n=1}^{\infty} \frac{\alpha+(1-\alpha)e^{-\lambda_n T}-1}{\alpha+(1-\alpha)e^{-\lambda_n T}}(v_n,\chi)v_n \tag{24}$$

$$+\sum_{n=1}^{\infty} \frac{\alpha e^{\mu_n T}+(1-\alpha)-e^{\mu_n T}}{\alpha e^{\mu_n T}+(1-\alpha)}(w_n,\chi)w_n \tag{25}$$

For all $\alpha \in [1/2, 1]$ the value

$$\frac{\alpha+(1-\alpha)e^{-\lambda_n T}-1}{\alpha+(1-\alpha)e^{-\lambda_n T}} = \frac{(\alpha-1)(1-e^{-\lambda_n T})}{\alpha+(1-\alpha)e^{-\lambda_n T}} \tag{26}$$

is nonnegative and nondecreasing with α since its derivative with respect to α is as follows:

$$\frac{1-e^{-\lambda_n T}}{(\alpha+(1-\alpha)e^{-\lambda_n T})^2} \geq 0$$

Consequently, the value (25) on the segment [1/2, 1] takes on the lowest value when $\alpha = 1/2$, i.e.

$$-1 \leq \frac{e^{-\lambda_n T}-1}{e^{-\lambda_n T}+1} \leq \frac{(\alpha-1)(1-e^{-\lambda_n T})}{\alpha+(1-\alpha)e^{-\lambda_n T}} \leq 0 \tag{27}$$

For all $\alpha \in [0, 1]$ the value

$$\frac{\alpha e^{\mu_n T}+(1-\alpha)-e^{\mu_n T}}{\alpha e^{\mu_n T}+(1-\alpha)} = \frac{(1-\alpha)(1-e^{\mu_n T})}{\alpha e^{\mu_n T}+(1-\alpha)} \tag{28}$$

is nonnegative and nonincreasing with α since its derivative with respect to α is as follows:

$$\frac{e^{\mu_n T}(e^{\mu_n T}-1)}{(\alpha e^{\mu_n T}+(1-\alpha))^2} \leq 0$$

Therefore, the value (27) is the greatest for $\alpha = 0$, i.e.

$$0 \leq \frac{(1-\alpha)(1-e^{\mu_n T})}{\alpha e^{\mu_n T}+(1-\alpha)} \leq 1-e^{\mu_n T} \leq 1 \tag{29}$$

Based on the inequalities (26) and (28), from (24) it is inferred that uniformly in $\alpha \in [1/2, 1]$ the following estimate is true:

$$\begin{aligned} \|\chi - u(0,\alpha)\|^2 &= \sum_{n=1}^{\infty} \frac{(\alpha + (1-\alpha)e^{-\lambda_n T} - 1)^2}{(\alpha + (1-\alpha)e^{-\lambda_n T})^2}(v_n, \chi)^2 \\ &+ \sum_{n=1}^{\infty} \frac{(\alpha e^{\mu_n T} + (1-\alpha) - e^{\mu_n T})^2}{(\alpha e^{\mu_n T} + (1-\alpha))^2}(w_n, \chi)^2 \\ &\leq \sum_{n=1}^{\infty}(v_n, \chi)^2 + \sum_{n=1}^{\infty}(w_n, \chi)^2 = \|\chi\|^2. \end{aligned} \tag{30}$$

Now we show that

$$\lim_{\alpha \to 1} \|u(0,\alpha) - \chi\| = 0, \quad \forall \chi \in M. \tag{31}$$

where M is a particular set dense in H. Then according to the Banach-Steinhaus theorem, (23) results from (29) and (30). In this case as a set M we take all χ representable in the form given below:

$$\chi = \sum_{n=1}^{\infty} c_n v_n + \sum_{n=1}^{N} d_n w_n, \quad \forall N < \infty.$$

Therefore,

$$\begin{aligned} \|\chi - u(0,\alpha)\|^2 &= \sum_{n=1}^{\infty} \frac{(\alpha - 1)^2(1 - e^{-\lambda_n T})^2}{(\alpha + (1-\alpha)e^{-\lambda_n T})^2}(v_n, \chi)^2 \\ &+ \sum_{n=1}^{N} \frac{(1-\alpha)^2(1 - e^{\mu_n T})^2}{(\alpha e^{\mu_n T} + (1-\alpha))^2}(w_n, \chi)^2 \\ &\leq \frac{(1-\alpha)^2}{\alpha^2}\sum_{n=1}^{\infty}(v_n, \chi)^2 + \frac{(1-\alpha)^2}{\alpha^2} e^{-2\mu_N T}\sum_{n=1}^{N}(w_n, \chi)^2 \\ &\leq \frac{(1-\alpha)^2}{\alpha^2} e^{-2\mu_N T}\|\chi\|^2 \end{aligned}$$

and for such χ the expressions (30) are true.

REFERENCES

[1] Brish N.I., Valeshkevith I.N. The Fourier Method for Nonstationary Equations with General Boundary Conditions. *Differential Equations* **1**(3), 1965, P. 393–399.