

Third-Order Generalized Discontinuous Impulsive Problems on the Half-Line

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Abstract. In this paper, we improve the existing results in the literature by presenting weaker sufficient conditions for the solvability of a third-order impulsive problem on the half-line, having generalized impulse effects. More precisely, our nonlinearities do not need to be positive nor sublinear and the monotone assumptions are local ones. Our method makes use of some truncation and perturbed techniques and on the equiconvergence at infinity and the impulsive points. The last section contains an application to a boundary layer flow problem over a stretching sheet with and without heat transfer.

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1 Introduction

In this work we deal with a boundary value problem composed of the third-order differential equation on the half real line

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad t \in [0, +\infty), \quad (1.1)$$

where $f : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function, together with the boundary conditions

$$u(0) = A, \quad u'(0) = B, \quad u''(+\infty) = C, \quad (1.2)$$

with $A, B, C \in \mathbb{R}$, $u''(+\infty) := \lim_{t \rightarrow +\infty} u''(t)$ and the impulsive effects given by the generalized functions

$$\begin{aligned} \Delta u(t_k) &= I_{0k}(t_k, u(t_k), u'(t_k), u''(t_k)), \\ \Delta u'(t_k) &= I_{1k}(t_k, u(t_k), u'(t_k), u''(t_k)), \\ \Delta u''(t_k) &= I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)), \end{aligned} \tag{1.3}$$

with $0 = t_0 < t_1 < t_2 < \dots < t_k < \dots$, $k \in \mathbb{N}$, such that $\lim_{k \rightarrow +\infty} t_k = +\infty$ and $I_{ik} : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ Carathéodory sequences for $i = 0, 1, 2$ and $k \in \mathbb{N}$.

We point out that the technique presented in this paper can be easily adapted, with obvious changes, to n^{th} order problems of the type

$$u^{(n)}(t) = f(t, u(t), \dots, u^{(n-1)}(t)), \quad t \in [0, +\infty), \tag{1.4}$$

$$u^{(i)}(0) = A_i, \quad u^{(n-1)}(+\infty) = B, \quad A_i, B \in \mathbb{R}, \quad i = 0, 1, \dots, n - 2,$$

$$\Delta u^{(j)}(t_k) = I_{ik}(t_k, u(t_k), \dots, u^{(n-1)}(t_k)), \quad j = 0, 1, \dots, n - 1, k \in \mathbb{N}. \tag{1.5}$$

The option for order three here, is due to clearance reasons, to highlight the method and not make the reading more difficult with heavy notation.

These higher-order boundary value problems with asymptotic conditions can model some real phenomena as gas pressure in a semi-infinite porous medium, draining or coating fluid-flow problems, and other evolution of physical processes. Likewise, they are useful in more theoretical studies such as on nonlinear elliptic equations, to prove the existence of radially symmetric solutions, or heteroclinic and homoclinic solutions of differential equations, or coupled systems of differential and integral equations. As related works we mention, for instance, [2, 5, 16, 17, 18, 19, 26, 27].

As the infinite interval is noncompact, the discussion about sufficient conditions for the solvability of boundary value problems on the half-line is more delicate. In the literature the main methods to obtain existence results are the extension of continuous solutions on finite intervals via a diagonalization process, lower and upper solutions and fixed point theorems in Banach weighted spaces (see [3, 4, 11, 30] and their references).

Impulsive problems, that is, situations where sudden variations happens, have had an important development in last decades, mostly due to their applicability to real life phenomena. See, for example, [1, 12, 13, 14, 16, 21, 24, 25, 28] and the references therein.

In [29], the authors consider a problem similar to (1.4)–(1.5) where the nonlinearity and all the impulsive functions must be sublinear and nondecreasing in all space variables. The existence of positive solutions is proved using cone theory and Mönch’s fixed point theorem, together with a monotone iterative technique.

Motivated by this work, we study problem (1.1)–(1.3) under weaker conditions, not only on the nonlinearity but also on the impulsive functions. Indeed, being more specific:

- the nonlinearity f is an L^1 - Carathéodory function, meaning that it could be discontinuous in time and, eventually, superlinear near the origin

or at $+\infty$. Moreover, there is not a monotone assumption on f in the highest order derivative, while in the other variables there is only the restriction of a local monotony in some strip;

- the impulsive functions I_{ik} , with $i = 0, 1, 2$, are locally monotone, that is, the monotonicity is required only on a strip. Moreover, the sequence I_{2k} has a different monotone behavior of [29] and there is no monotone assumption at all on the highest order variable;
- the solutions may have negative values.

Our method relies on the lower and upper solutions technique, which reveals to be adequate to these impulsive boundary value problems, adding to the existence of solution its localization and some qualitative data on its behavior as well. We apply some truncation and perturbation techniques suggested, for example, in [6, 9, 10], together with equiconvergence at $+\infty$ and on the impulsive points, as it appears in [8].

The paper is organized in the following way: Section 2 contains the definition of the Banach spaces, the corresponding weighted norms, and other auxiliary results as well. In Section 3 we present the main theorem: an existence and localization result, where we prove the existence of at least one solution, and some bounds on the first and second derivatives. The last section, has an application to a boundary layer flow problem over a stretching sheet with and without heat transfer.

2 Definitions and auxiliary results

A key argument of our method is based on a weighted space with some asymptotic assumptions.

Consider the spaces

$$PC^2[0, +\infty] = \left\{ \begin{array}{l} u : u \in C^2([0, +\infty]; \mathbb{R}) \text{ for } t \neq t_k, u^{(i)}(t_k) = u^{(i)}(t_k^-), \\ u^{(i)}(t_k^+) \text{ exists for } k = 1, 2, \dots, m, \text{ and } i = 0, 1, 2 \end{array} \right\}$$

and

$$X = \left\{ x \in PC^2[0, +\infty) : \lim_{t \rightarrow +\infty} \frac{x^{(i)}(t)}{w_i(t)} \text{ exists, } i = 0, 1, 2 \right\}$$

with $w_i(t) = 1 + t^{2-i}$ and the norm $\|y\| = \max \{\|y\|_0, \|y\|_1, \|y\|_2\}$, where

$$\|y\|_0 = \sup_{0 \leq t < +\infty} \left\{ \frac{|y(t)|}{1 + t^2} \right\}, \|y\|_1 = \sup_{0 \leq t < +\infty} \left\{ \frac{|y'(t)|}{1 + t} \right\}, \|y\|_2 = \sup_{0 \leq t < +\infty} \left\{ \frac{|y''(t)|}{2} \right\}.$$

Therefore, $(X, \|\cdot\|)$ is a Banach space.

The nonlinearities will have the regularity of L^1 - Carathéodory functions defined as it follows:

DEFINITION 1. A function $f : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is L^1 - Carathéodory if it satisfies

- i) for each $(x, y, z) \in \mathbb{R}^3$, $t \mapsto f(t, x, y, z)$ is measurable on $[0, +\infty)$;
- ii) for almost every $t \in [0, +\infty)$, $(x, y, z) \mapsto f(t, x, y, z)$ is continuous in \mathbb{R}^3 ;
- iii) for each $\rho > 0$, there exists a positive function $\psi_\rho \in L^1[0, +\infty)$ such that, for $\max\{\|x\|_0, \|y\|_1, \|z\|_2\} < \rho$,

$$|f(t, x, y, z)| \leq \psi_\rho(t), \text{ a.e. } t \in [0, +\infty).$$

The impulsive effects are given in terms of sequences of functions as in next definition.

DEFINITION 2. A sequence $(w_n)_{n \in \mathbb{N}}$ is a Carathéodory sequence if

- (i) for each $(x, y, z) \in \mathbb{R}^3$, $(x, y, z) \rightarrow w_n(x, y, z)$ is continuous for all $n \in \mathbb{N}$;
- (ii) for each $\rho > 0$, there are nonnegative constants $\lambda_{n\rho} \geq 0$ with $\sum_{n=1}^{+\infty} \lambda_{n\rho} < +\infty$ such that for $|x| < \rho(1 + t^2)$, $|y| < \rho(1 + t)$, $|z| < 2\rho$, for $t \in [0, +\infty)$, we have

$$|w_n(x, y, z)| \leq \lambda_{n\rho}, \text{ for every } n \in \mathbb{N}.$$

Next lemma gives the exact solution for the associated linear and homogeneous problem:

Lemma 1. *If $e \in L^1[0, +\infty)$, then the boundary value problem*

$$\begin{cases} u'''(t) = e(t), & t \in (0, +\infty), \\ u(0) = A, \quad u'(0) = B, \quad u''(+\infty) = C \end{cases} \tag{2.1}$$

has a unique solution in X . Moreover, this solution can be expressed as

$$\begin{aligned} u(t) = & A + Bt + \frac{Ct^2}{2} + \sum_{k: t > t_k} \\ & \times \left[I_{0,k}(t_k, u(t_k), u'(t_k), u''(t_k)) + I_{1,k}(t_k, u(t_k), u'(t_k), u''(t_k))(t - t_k) \right. \\ & \left. + I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)) \frac{(t - t_k)^2}{2} \right] \\ & - \frac{t^2}{2} \sum_{k=1}^{+\infty} I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)) - \int_0^{+\infty} G(t, s) e(s) ds, \end{aligned}$$

where $G(t, s)$ is the Green function of the homogeneous problem associated to (2.1), given by

$$G(t, s) = \begin{cases} \frac{1}{2}s^2 - st, & 0 \leq s \leq t, \\ -\frac{1}{2}t^2, & t \leq s \leq +\infty. \end{cases} \tag{2.2}$$

The proof follows from standard integrations and usual arguments and is omitted.

The following theorem, to be used forward, gives a general criterion for relative compactness:

Theorem 1. (*[8]*) Let $M \subset C_\infty = \{x \in C[0, +\infty) : \lim_{t \rightarrow +\infty} x(t) \text{ exists}\}$. Then M is relatively compact if the following conditions hold:

1. All functions in M are uniformly bounded;
2. All functions in M are equicontinuous on any compact interval of $[0, +\infty)$;
3. All functions in M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a t_ϵ such that $|x(t) - x(+\infty)| < \epsilon$, for all $t > t_\epsilon$ and $x \in M$.

3 Main result

In this section we prove the existence of at least one solution for the problem (1.1)–(1.3), applying lower and upper solutions method. Moreover, some data on its behavior and variation are given.

First we define lower and upper functions for impulsive problems.

DEFINITION 3. Given $A, B, C \in \mathbb{R}$, a function $\alpha \in X$ is a lower solution of problem (1.1)–(1.3) if

$$\left\{ \begin{array}{l} \alpha'''(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t)), t \in [0, +\infty), \\ \alpha(0) \leq A, \quad \alpha'(0) \geq B, \quad \alpha''(+\infty) \leq C, \\ \Delta\alpha(t_k) \leq I_{0k}(t_k, \alpha(t_k), \alpha'(t_k), \alpha''(t_k)), \\ \Delta\alpha'(t_k) > I_{1k}(t_k, \alpha(t_k), \alpha'(t_k), \alpha''(t_k)), \\ \Delta\alpha''(t_k) > I_{2k}(t_k, \alpha(t_k), \alpha'(t_k), \alpha''(t_k)), \end{array} \right.$$

with $k \in \mathbb{N}$.

A function $\beta \in X$ is an upper solution if it verifies the reversed inequalities.

Forward, the following assumption will play a key role:

(A) There is $\xi > 0$ such that

$$\xi \geq \max \left\{ \begin{array}{l} \|\alpha\|_0, \|\beta\|_0, \|\alpha'\|_1, \|\beta'\|_1, \|\alpha''\|_2, \|\beta''\|_2, \\ |A| + \frac{|B|+|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{0k\xi} + \sum_{k=1}^{+\infty} \lambda_{1k\xi} + \sum_{k=1}^{+\infty} \lambda_{2k\xi} \\ \quad + M_0 \left(\frac{\pi}{2} + \int_0^{+\infty} \psi_\xi(s) ds \right), \\ |B| + |C| + \sum_{k=1}^{+\infty} \lambda_{1k\xi} + 2 \sum_{k=1}^{+\infty} \lambda_{2k\xi} \\ \quad + M_1 \left(\frac{\pi}{2} + \int_0^{+\infty} \psi_\xi(s) ds \right), \\ \frac{|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{2k\xi} + \frac{1}{2} \int_0^{+\infty} \psi_\xi(s) ds + \frac{\pi}{4} \end{array} \right\},$$

where, for Catathéodory sequences $I_{ik} : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$, $i = 0, 1, 2$, $k \in \mathbb{N}$,

$$|I_{ik}(t_k, y_0, y_1, y_2)| \leq \sum_{k=1}^{+\infty} \lambda_{ik\xi} < +\infty,$$

when

$$|y_0| < \xi(1 + t^2), |y_1| < \xi(1 + t), |y_2| < 2\xi, \text{ for } t \in [0, +\infty),$$

$$M_0 := \sup_{t \in [0, +\infty[} \frac{|G(t, s)|}{1 + t^2}, M_1 := \sup_{t \in [0, +\infty[} \frac{|\frac{\partial G}{\partial t}(t, s)|}{1 + t},$$

for $s \in [0, +\infty)$, and

$$|f(t, x, y, z)| \leq \psi_\xi(t), \text{ a.e. } t \in [0, +\infty),$$

when $\max \{\|x\|_0, \|y\|_1, \|z\|_2\} < \xi$.

The existence and localization result is given by next theorem.

Theorem 2. Consider $A, B, C \in \mathbb{R}$. Assume that there are α and β lower and upper solutions of problem (1.1)–(1.3) such that

$$\alpha''(t) \leq \beta''(t), \quad \forall t \in [0, +\infty). \quad (3.1)$$

Let $f : [0, +\infty) \times \mathbb{R}^4 \rightarrow \mathbb{R}$ be a L^1 – Carathéodory function with

$$f(t, \alpha(t), \alpha'(t), y_2) \geq f(t, y_0, y_1, y_2) \geq f(t, \beta(t), \beta'(t), y_2) \quad (3.2)$$

for $t \in [0, +\infty)$, $\alpha(t) \leq y_0 \leq \beta(t)$, $\alpha'(t) \leq y_1 \leq \beta'(t)$, and $y_2 \in \mathbb{R}$.

Assume that $I_{ik} : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ are Carathéodory sequences, for $i = 0, 1, 2$, $k \in \mathbb{N}$, such that

$$I_{0k}(t_k, \alpha(t_k), \alpha'(t_k), \alpha''(t_k)) \leq I_{0k}(t_k, y_0, y_1, y_2) \leq I_{0k}(t_k, \beta(t_k), \beta'(t_k), \beta''(t_k)),$$

for $\alpha^{(i)}(t_k) \leq y_i \leq \beta^{(i)}(t_k)$, $i = 0, 1, 2$, (3.3)

$$I_{1k}(t_k, \alpha(t_k), \alpha'(t_k), \alpha''(t_k)) \leq I_{1k}(t_k, y_0, y_1, y_2) \leq I_{1k}(t_k, \beta(t_k), \beta'(t_k), \beta''(t_k)),$$

for $\alpha^{(i)}(t_k) \leq y_i \leq \beta^{(i)}(t_k)$, $i = 0, 1, 2$, (3.4)

$$I_{2k}(t_k, \beta(t_k), \beta'(t_k), y_2) \leq I_{2k}(t_k, y_0, y_1, y_2) \leq I_{2k}(t_k, \alpha(t_k), \alpha'(t_k), y_2),$$

for $\alpha^{(i)}(t_k) \leq y_i \leq \beta^{(i)}(t_k)$, $i = 0, 1, y_2 \in \mathbb{R}$. (3.5)

If there is $\xi > 0$ such that assumption (A) holds, then there is at least $u(t) \in X$, a solution of (1.1)–(1.3), such that

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \quad \forall t \in [0, +\infty), \quad i = 0, 1, 2.$$

Proof. Let $\alpha, \beta \in X$ be, respectively, lower and upper solutions of (1.1)–(1.3) verifying (3.1). Notice that the relations $\alpha(t) \leq \beta(t)$ and $\alpha'(t) \leq \beta'(t)$, $\forall t \in [0, +\infty)$, are obtained by integration from (3.1) and the boundary conditions (1.2). Consider the modified and perturbed problem composed of the differential equation

$$u'''(t) = f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \delta_2(t, u''(t))) + \frac{1}{1+t^2} \frac{u''(t) - \delta_2(t, u''(t))}{1 + |u''(t) - \delta_2(t, u''(t))|}, \quad (3.6)$$

for $t \in [0, +\infty)$, where the functions $\delta_j : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$, $j = 0, 1, 2$ are given by

$$\delta_j(t, u^{(j)}(t)) = \begin{cases} \beta^{(j)}(t), & u^{(j)}(t) > \beta^{(j)}(t), \\ u^{(j)}(t), & \alpha^{(j)}(t) \leq u^{(j)}(t) \leq \beta^{(j)}(t), \\ \alpha^{(j)}(t), & u^{(j)}(t) < \alpha^{(j)}(t), \end{cases}$$

the boundary conditions (1.2) and the truncated impulsive effects

$$\Delta u^{(j)}(t_k) = I_{j,k}(t_k, \delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)), \delta_2(t_k, u''(t_k))), \quad j=0, 1, 2. \quad (3.7)$$

For clarity we divide the proof into claims.

CLAIM 1: *Problem (3.6), (1.2), (3.7) has at least one solution.*

Define the operator $\mathcal{T} : X \rightarrow X$ by

$$\begin{aligned} \mathcal{T}u(t) &= A + Bt + \frac{Ct^2}{2} + \sum_{k : t > t_k} \\ &\times \left[I_{0,k}(t_k, u(t_k), u'(t_k), u''(t_k)) + I_{1,k}(t_k, u(t_k), u'(t_k), u''(t_k)) (t - t_k) \right. \\ &\quad \left. + I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)) \frac{(t - t_k)^2}{2} \right] \\ &- \frac{t^2}{2} \sum_{k=1}^{+\infty} I_{2k}(t_k, u(t_k), u'(t_k), u''(t_k)) - \int_0^{+\infty} G(t, s) F(u(s)) ds, \end{aligned}$$

with $G(t, s)$ given by (2.2), and

$$\begin{aligned} F(u(s)) &:= f(s, \delta_0(s, u(s)), \delta_1(s, u'(s)), \delta_2(s, u''(s))) \\ &\quad + \frac{1}{1 + t^2} \frac{u''(s) - \delta_2(t, u''(s))}{1 + |u''(s) - \delta_2(s, u''(s))|}. \end{aligned}$$

By Lemma 1, the fixed points of \mathcal{T} are solutions of the problem (3.6), (1.2) and (3.7). So it is enough to prove that \mathcal{T} has a fixed point.

For convenience we denote

$$I_{i,k} := I_{i,k}(t_k, u(t_k), u'(t_k), u''(t_k)), \quad \text{for } i = 0, 1, 2.$$

(1) $\mathcal{T} : X \rightarrow X$ is well defined.

Take

$$\rho > \max \{ \|\alpha\|_0, \|\beta\|_0, \|\alpha'\|_1, \|\beta'\|_1, \|\alpha''\|_2, \|\beta''\|_2 \}. \quad (3.8)$$

As f is an L^1 -Carathéodory function, by Definition 1, for $u \in X$ with $\|u\| < \rho$,

$$\int_0^{+\infty} |F(u(s))| ds \leq \int_0^{+\infty} \left(\psi_\rho(s) + \frac{1}{1 + t^2} \right) ds \leq \int_0^{+\infty} \psi_\rho(s) ds + \frac{\pi}{2} < +\infty,$$

and so $F(u(s)) \in L^1([0, +\infty[)$.

By Lebesgue Dominated Theorem and Definition 2,

$$\lim_{t \rightarrow +\infty} \frac{|\mathcal{T}u(t)|}{1 + t^2} \leq \frac{|C|}{2} + \sum_{k=1}^{+\infty} |I_{2k}| \leq \frac{|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{2k\rho} < +\infty.$$

Analogously,

$$\lim_{t \rightarrow +\infty} \frac{|\mathcal{T}u'(t)|}{1 + t} \leq |C| + 2 \sum_{k=1}^{+\infty} |I_{2k}| \leq |C| + 2 \sum_{k=1}^{+\infty} \lambda_{2k\rho} < +\infty,$$

and

$$\lim_{t \rightarrow +\infty} \frac{|(\mathcal{T}u)''(t)|}{2} \leq \frac{1}{2} \left(|C| + 2 \sum_{k=1}^{+\infty} \lambda_{2k\rho} + \int_0^{+\infty} \psi_\rho(s) ds \right) < +\infty.$$

Therefore, $\mathcal{T}u \in X$.

(2) \mathcal{T} is continuous.

For any convergent sequence $u_n \rightarrow u$ in X , there exists $r_1 > 0$ such that, for $\|u_n\| < r_1$, we have

$$\begin{aligned} \|\mathcal{T}u_n - \mathcal{T}u\| &= \max \{ \|\mathcal{T}u_n - \mathcal{T}u\|_0, \|(\mathcal{T}u_n)' - (\mathcal{T}u)'\|_1, \|(\mathcal{T}u_n)'' - (\mathcal{T}u)''\|_2 \} \\ &\leq \int_0^{+\infty} \max \{ M_0, M_1 \} |F(u_n(s)) - F(u(s))| ds \\ &\leq \int_0^{+\infty} |F(u_n(s)) - F(u(s))| ds \rightarrow 0, \quad n \rightarrow +\infty. \end{aligned}$$

(3) \mathcal{T} is compact.

Let $D \subset X$ be any bounded subset. Therefore there is $R > 0$ such that $\|u\| < R, \forall u \in D$. Then,

$$\begin{aligned} \|\mathcal{T}u\|_0 &= \sup_{t \in [0, +\infty[} \frac{|\mathcal{T}u(t)|}{1+t^2} \leq |A| + \frac{|B|+|C|}{2} + \sum_{k=1}^{+\infty} |I_{0k}| \\ &\quad + \sum_{k=1}^{+\infty} |I_{1k}| + \sum_{k=1}^{+\infty} |I_{2k}| + \int_0^{+\infty} \sup_{t \in [0, +\infty[} \frac{|G(t,s)|}{1+t^2} \psi_R(s) ds \\ &\leq |A| + \frac{|B|+|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{0kR} + \sum_{k=1}^{+\infty} \lambda_{1kR} + \sum_{k=1}^{+\infty} \lambda_{2kR} \\ &\quad + M_0 \left(\frac{\pi}{2} + \int_0^{+\infty} \psi_R(s) ds \right) < +\infty, \end{aligned}$$

$$\begin{aligned} \|\mathcal{T}u\|_1 &= \sup_{t \in [0, +\infty[} \frac{|(\mathcal{T}u(t))'|}{1+t} \\ &\leq |B| + |C| + \sup_{t \in [0, +\infty[} \frac{1}{1+t} \sum_{k:t > t_k} |I_{1k}| + \sup_{t \in [0, +\infty[} \sum_{k:t > t_k} |I_{2k}| \frac{t-t_k}{1+t} \\ &\quad + \sup_{t \in [0, +\infty[} \frac{t}{1+t} \sum_{k=1}^{+\infty} |I_{2k}| + \int_0^{+\infty} M_1 |F(u(s))| ds \\ &\leq |B| + |C| + \sum_{k=1}^{+\infty} \lambda_{1kR} + 2 \sum_{k=1}^{+\infty} \lambda_{2kR} + M_1 \left(\frac{\pi}{2} + \int_0^{+\infty} \psi_p(s) ds \right) < +\infty, \end{aligned}$$

and

$$\|\mathcal{T}u\|_2 = \sup_{t \in [0, +\infty[} \frac{|(\mathcal{T}u(t))''|}{2} \leq \frac{|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{2kR} + \frac{1}{2} \int_0^{+\infty} \psi_R(s) ds + \frac{\pi}{4} < +\infty.$$

So \mathcal{T} is uniformly bounded. Moreover, \mathcal{T} is equicontinuous on each interval $(t_k, t_{k+1}]$. To see this, let $t_1, t_2 \in (t_k, t_{k+1}]$, with $t_1 < t_2$; as $t_1 \rightarrow t_2$,

$$\begin{aligned} \left| \frac{\mathcal{T}u(t_1)}{1+t_1^2} - \frac{\mathcal{T}u(t_2)}{1+t_2^2} \right| &\leq \left| \frac{Bt_1 + \frac{C}{2}t_1^2}{1+t_1^2} - \frac{Bt_2 + \frac{C}{2}t_2^2}{1+t_2^2} \right| \\ &+ \left| \sum_{k : t_1 > t_k} \left[I_{0k} + I_{1k}(t_1 - t_k) + I_{2k} \frac{(t_1 - t_k)^2}{2} \right] \right. \\ &\quad \left. - \sum_{k : t_2 > t_k} \left[I_{0k} + I_{1k}(t_2 - t_k) + I_{2k} \frac{(t_2 - t_k)^2}{2} \right] \right| + \frac{1}{2} |t_1^2 - t_2^2| \sum_{k=1}^{+\infty} |I_{2k}| \\ &+ \int_0^{+\infty} \left| \frac{G(t_1, s)}{1+t_1^2} - \frac{G(t_2, s)}{1+t_2^2} \right| (\psi_R(s) + 1) ds \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \left| \frac{(\mathcal{T}u)'(t_1)}{1+t_1} - \frac{(\mathcal{T}u)'(t_2)}{1+t_2} \right| &\leq \left| \frac{Ct_1}{1+t_1} - \frac{Ct_2}{1+t_2} \right| + |t_1 - t_2| \sum_{k=1}^{+\infty} |I_{2k}| \\ &+ \left| \sum_{k : t_1 > t_k} [I_{1k} + I_{2k}(t_1 - t_k)] - \sum_{k : t_2 > t_k} [I_{1k} + I_{2k}(t_2 - t_k)] \right| \\ &+ \int_0^{+\infty} \left| \frac{\partial G}{\partial t}(t_1, s) - \frac{\partial G}{\partial t}(t_2, s) \right| (\psi_R(s) + 1) ds \longrightarrow 0. \end{aligned}$$

The function $\frac{\partial^2 G}{\partial t^2}(t, s)$ is not continuous for $s = t$ but the jump is controlled by 1. Then,

$$\begin{aligned} \left| \frac{(\mathcal{T}u)''(t_1)}{2} - \frac{(\mathcal{T}u)''(t_2)}{2} \right| &\leq \frac{1}{2} \sum_{k : t_1 < t_k < t_2} |I_{2k}| \\ &+ \left| \int_{t_1}^{+\infty} F(u(s)) ds - \int_{t_2}^{+\infty} F(u(s)) ds \right| \\ &\leq \frac{1}{2} \sum_{k : t_1 < t_k < t_2} |I_{2k}| + \int_{t_1}^{t_2} (\psi_R(s) + 1) ds \longrightarrow 0 \text{ as } t_1 \rightarrow t_2. \end{aligned}$$

To prove that $\mathcal{T}D$ is equiconvergent at infinity we apply, as $t \rightarrow +\infty$,

$$\begin{aligned} \left| \frac{\mathcal{T}u(t)}{1+t^2} - \lim_{t \rightarrow +\infty} \frac{\mathcal{T}u(t)}{1+t^2} \right| &\leq \left| \frac{A+Bt}{1+t^2} + \frac{Ct^2}{2+2t^2} - \frac{C}{2} \right| \\ &+ \frac{1}{1+t^2} \left| \sum_{k : t > t_k} \left[I_{0k} + I_{1k}(t_1 - t_k) + I_{2k} \frac{(t_1 - t_k)^2}{2} \right] \right. \\ &\quad \left. - \frac{1}{2} \sum_{k=1}^{+\infty} I_{2k} \right| \\ &+ \left| \frac{1}{2} \frac{t^2}{1+t^2} \sum_{k=1}^{+\infty} |I_{2k}| - \frac{1}{2} \sum_{k=1}^{+\infty} |I_{2k}| \right| \\ &+ \int_0^{+\infty} \left| \frac{G(t, s)}{1+t^2} + \frac{1}{2} \right| (\psi_R(s) + 1) ds \longrightarrow 0, \end{aligned}$$

$$\begin{aligned}
 & \left| \frac{(\mathcal{T}u)'(t)}{1+t} - \lim_{t \rightarrow +\infty} \frac{(\mathcal{T}u)'(t)}{1+t} \right| \leq \left| \frac{B+Ct}{1+t} - C \right| \\
 & \quad + \frac{1}{1+t} \left| \sum_{k:t>t_k} [I_{1k} + I_{2k}(t_1 - t_k)] - \sum_{k=1}^{+\infty} I_{2k} \right| \\
 & \quad + \left| \frac{t}{1+t} \sum_{k=1}^{+\infty} |I_{2k}| - \sum_{k=1}^{+\infty} |I_{2k}| \right| \\
 & \quad + \int_0^{+\infty} \left| \frac{\partial G}{\partial t}(t, s)}{1+t} + 1 \right| (\psi_R(s) + 1) ds \longrightarrow 0, \\
 & \left| \frac{(\mathcal{T}u)''(t)}{2} - \lim_{t \rightarrow +\infty} \frac{(\mathcal{T}u)''(t)}{2} \right| \leq \frac{1}{2} \left| \sum_{k:t>t_k} I_{2k} - \sum_{k=1}^{+\infty} I_{2k} \right| \\
 & \quad + \int_0^{+\infty} \left| \frac{\partial^2 G}{\partial t^2}(t, s) + 1 \right| (\psi_R(s) + 1) ds \longrightarrow 0, \text{ as } t \rightarrow +\infty.
 \end{aligned}$$

Finally, to prove that $\mathcal{T}D$ is equiconvergent at the impulsive moments we apply, as $t \rightarrow t_i^+$, for $i \in \mathbb{N}$,

$$\begin{aligned}
 & \left| \frac{\mathcal{T}u(t)}{1+t^2} - \lim_{t \rightarrow t_i^+} \frac{\mathcal{T}u(t)}{1+t^2} \right| \leq \left| \frac{Bt + \frac{C}{2}t^2}{1+t^2} - \frac{Bt_i + \frac{C}{2}t_i^2}{1+t_i^2} \right| \\
 & \quad + \left| \frac{1}{1+t^2} \sum_{k:t>t_k} \left(I_{0k} + I_{1k}(t - t_k) + I_{2k} \frac{(t-t_k)^2}{2} \right) \right. \\
 & \quad \left. - \frac{1}{1+t_i^2} \sum_{k:t_i^+>t_k} \left(I_{0k} + I_{1k}(t_i^+ - t_k) + I_{2k} \frac{(t_i^+ - t_k)^2}{2} \right) \right| \\
 & \quad + \left| \left(-\frac{t^2}{1+t^2} + \frac{t_i^2}{1+t_i^2} \right) \sum_{k=1}^{+\infty} I_{2k} \right| \\
 & \quad + \int_0^{+\infty} \left| \frac{G(t, s)}{1+t^2} - \frac{G(t_i^+, s)}{1+t_i^2} \right| (\psi_R(s) + 1) ds \longrightarrow 0,
 \end{aligned}$$

uniformly on $u \in D$, as $t \rightarrow t_i^+$,

$$\begin{aligned}
 & \left| \frac{(\mathcal{T}u)'(t)}{1+t} - \lim_{t \rightarrow t_i^+} \frac{(\mathcal{T}u)'(t)}{1+t} \right| \leq \left| \frac{Ct}{1+t} - \frac{Ct_i}{1+t_i} \right| \\
 & \quad + \left| \frac{1}{1+t} \sum_{k:t>t_k} [I_{1k} + I_{2k}(t - t_k)] - \frac{1}{1+t_i} \sum_{k:t_i^+>t_k} [I_{1k} + I_{2k}(t_i - t_k)] \right| \\
 & \quad + \left| \left(-\frac{t}{1+t} + \frac{t_i}{1+t_i} \right) \sum_{k=1}^{+\infty} I_{2k} \right| \\
 & \quad + \int_0^{+\infty} \left| \frac{\partial G}{\partial t}(t, s)}{1+t^2} - \frac{\partial G}{\partial t}(t_i^+, s)}{1+t_i^2} \right| (\psi_R(s) + 1) ds \longrightarrow 0, \text{ as } t \rightarrow t_i^+, \\
 & \left| \frac{(\mathcal{T}u)''(t)}{2} - \lim_{t \rightarrow t_i^+} \frac{(\mathcal{T}u)''(t)}{2} \right| \leq \left| \frac{1}{2} \sum_{k:t>t_k} I_{2k} - \frac{1}{2} \sum_{k:t_i^+>t_k} I_{2k} \right|
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \left| \int_t^{+\infty} F(u(s)) ds - \int_{t_i}^{+\infty} F(u(s)) ds \right| \\
 & \leq \frac{1}{2} \left| \sum_{k : t > t_k} I_{2k} - \sum_{k : t_i^+ > t_k} I_{2k} \right| + \frac{1}{2} \left| \int_{t_i}^t (\psi_R(s) + 1) ds \right| \longrightarrow 0,
 \end{aligned}$$

uniformly for $u \in D$, as $t \longrightarrow t_i^+$.

So, by Theorem 1, $\mathcal{T}D$ is relatively compact. To apply Schauder’s Fixed Point Theorem, we need to show that $\mathcal{T} : D \rightarrow D$.

CLAIM 2: *For some nonempty, closed, bounded and convex subset $D \subset X$, $\mathcal{T}D \subseteq D$.*

By assumption (A) in Step (3) of the previous Claim 1, take $R > 0$ such that

$$R \geq \max \left\{ \begin{array}{l} \rho, |A| + \frac{|B|+|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{0kR} + \sum_{k=1}^{+\infty} \lambda_{1kR} + \sum_{k=1}^{+\infty} \lambda_{2kR} \\ \quad + M_0 \left(\frac{\pi}{2} + \int_0^{+\infty} \psi_R(s) ds \right), \\ |B| + |C| + \sum_{k=1}^{+\infty} \lambda_{1kR} + 2 \sum_{k=1}^{+\infty} \lambda_{2kR} \\ \quad + M_1 \left(\frac{\pi}{2} + \int_0^{+\infty} \psi_R(s) ds \right), \\ \frac{|C|}{2} + \sum_{k=1}^{+\infty} \lambda_{2kR} + \frac{1}{2} \int_0^{+\infty} \psi_R(s) ds + \frac{\pi}{4} \end{array} \right\},$$

with ρ given by (3.8).

From the calculations in Claim 1, for every $u \in D$ such that $\|u\| < R$, we have $\mathcal{T}D \subseteq D$. Hence \mathcal{T} is completely continuous, by Schauder’s Fixed Point Theorem, \mathcal{T} has at least one fixed point $u \in X$.

From Lemma 1, (3.6) and (3.7), this fixed point will be a solution of the problem (1.1)–(1.3) if

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \quad i = 0, 1, 2, \quad \forall t \in [0, +\infty).$$

CLAIM 3: *Every solution of problem (3.6), (2), (3.7), satisfies*

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \quad i = 0, 1, 2, \quad \forall t \in [0, +\infty).$$

Let u be a solution of problem (3.6), (2), (3.7). Suppose for a contradiction that there is $u''(t) < \alpha''(t)$, and define

$$\inf_{t \in [0, +\infty[} u''(t) - \alpha''(t) := u''(t_*) - \alpha''(t_*) < 0.$$

Note that $t_* \neq +\infty$, as by (2) and Definition 3, $u''(+\infty) - \alpha''(+\infty) \geq 0$.

If $t_* = 0$, the following contradiction holds, by (3.2) and Definition 3:

$$\begin{aligned}
 0 & \leq u'''(0) - \alpha'''(0) = f(0, \delta_0(0, u(0)), \delta_1(0, u'(0)), \delta_2(0, u''(0))) \\
 & + \frac{u''(0) - \alpha''(0)}{1 + |u''(0) - \alpha''(0)|} - \alpha'''(0) < f(0, \delta_0(0, u(0)), \delta_1(0, u'(0)), \delta_2(0, u''(0))) \\
 & - \alpha'''(0) \leq f(0, \alpha(0), \alpha'(0), \alpha''(0)) - \alpha'''(0) \leq 0.
 \end{aligned}$$

Therefore $t_* \neq 0$.

Consider now that t_* is between two consecutive impulses. That is, there is a $p \in \mathbb{N}$ such that $t_* \in (t_p, t_{p+1})$. Then,

$$u'''(t_*) = \alpha'''(t_*), \quad u''(t_*) - \alpha''(t_*) < 0$$

and we have, by (3.2) and Definition 3, the contradiction

$$\begin{aligned} 0 &= u'''(t_*) - \alpha'''(t_*) = f(t_*, \delta_0(t_*, u(t_*)), \delta_1(t_*, u'(t_*)), u''(t_*)) \\ &\quad + \frac{1}{1+t_*^2} \frac{u''(t_*) - \delta_2(t_*, u(t_*))}{|u''(t_*) - \delta_2(t_*, u(t_*))| + 1} - \alpha'''(t_*) \\ &\leq f(t_*, \alpha(t_*), \alpha'(t_*), \alpha''(t_*)) + \frac{1}{1+t_*^2} \frac{u''(t_*) - \alpha''(t_*)}{|u''(t_*) - \alpha''(t_*)| + 1} - \alpha'''(t_*) \\ &< f(t_*, \alpha(t_*), \alpha'(t_*), \alpha''(t_*)) - \alpha'''(t_*) \leq 0. \end{aligned}$$

Assume now that the infimum is attained at an impulsive moment. So, we have two cases: $t_* = t_q^-$ or $t_* = t_q^+$. Firstly, consider that there is $q \in \mathbb{N}$ where

$$\min_{t \in [0, +\infty)} (u''(t) - \alpha''(t)) := u''(t_q) - \alpha''(t_q) < 0.$$

Then this contradiction holds:

$$\begin{aligned} 0 &\leq \Delta(u'' - \alpha'')(t_q) \\ &= I_{2,q}(t_q, \delta_0(t_q, u(t_q)), \delta_1(t_q, u'(t_q)), \delta_2(t_q, u''(t_q))) - \Delta\alpha''(t_q) \\ &= I_{2,q}(t_q, \delta_0(t_q, u(t_q)), \delta_1(t_q, u'(t_q)), \alpha''(t_q)) - \Delta\alpha''(t_q) \\ &\leq I_{2,q}(t_q, \alpha(t_q), \alpha'(t_q), \alpha''(t_q)) - \Delta\alpha''(t_q) < 0. \end{aligned}$$

In the second case, assume that

$$\inf_{t \in [0, +\infty)} u''(t) - \alpha''(t) := u''(t_*^+) - \alpha''(t_*^+) < 0.$$

Consider $\varepsilon > 0$ small enough such that

$$(u'' - \alpha'')(t) < 0, \quad u'''(t^+) - \alpha'''(t^+) \geq 0, \quad \text{for } t \in (t_q, t_q + \varepsilon).$$

So, for $t \in (t_q, t_q + \varepsilon)$, a contradiction can be obtained following the same arguments as for $t_* \in (t_p, t_{p+1})$. Therefore,

$$\alpha''(t) \leq u''(t), \quad \text{for } t \in [0, +\infty).$$

By a similar technique, it can be proved that $u''(t) \leq \beta''(t)$, for $t \in (0, +\infty)$, and then

$$\alpha''(t) \leq u''(t) \leq \beta''(t), \quad \text{for } t \in [0, +\infty). \quad (3.9)$$

Integrating the first inequality of (3.9) for $t \in [0, t_1]$, by (1.2) and Definition 3,

$$\alpha'(t) \leq u'(t) + \alpha'(0) - u'(0) \leq u'(t). \quad (3.10)$$

By integration in $(t_1, +\infty)$, (3.5), (3.10) and Definition 3, we have for $t \in (t_1, +\infty)$

$$\begin{aligned} \alpha'(t) &\leq u'(t) + \alpha'(t_1^+) - u'(t_1^+) = u'(t) + \alpha'(t_1^+) \\ &\quad - I_{11}(t_1, \delta_0(t_1, u(t_1)), \delta_1(t_1, u'(t_1)), \delta_2(t_1, u''(t_1))) - u'(t_1) \\ &\leq u'(t) + I_{11}(t_1, \alpha(t_1), \alpha'(t_1), \alpha''(t_1)) + \alpha'(t_1) \\ &\quad - I_{11}(t_1, \delta_0(t_1, u(t_1)), u'(t_1), u''(t_1)) - u'(t_1) \\ &\leq u'(t) + I_{11}(t_1, \alpha(t_1), \alpha'(t_1), \alpha''(t_1)) - I_{11}(t_1, \delta_0(t_1, u(t_1)), u'(t_1), u''(t_1)) \\ &\leq u'(t). \end{aligned}$$

Analogously, one can show that $u'(t) \leq \beta'(t), \forall t \in [0, +\infty)$ and, then,

$$\alpha'(t) \leq u'(t) \leq \beta'(t), \text{ for } t \in [0, +\infty). \tag{3.11}$$

Integrating the first inequality of (3.11) on $[0, t_1]$, we have

$$\alpha(t) \leq u(t) - u(0) + \alpha(0) \leq u(t),$$

and on $(t_1, +\infty)$, by (3.3) and Definition 3,

$$\begin{aligned} \alpha(t) &\leq u(t) + \alpha(t_1^+) - u(t_1^+) \leq u(t) + I_{01}(t_1, \alpha(t_1), \alpha'(t_1), \alpha''(t_1)) \\ &\quad + \alpha(t_1) - I_{01}(t_1, \delta_0(t_1, u(t_1)), \delta_1(t_1, u'(t_1)), \delta_2(t_1, u''(t_1))) - u(t_1) \\ &\leq u(t) + I_{01}(t_1, \alpha(t_1), \alpha'(t_1), \alpha''(t_1)) - I_{01}(t_1, u(t_1), u'(t_1), u''(t_1)) \leq u(t). \end{aligned}$$

So, $\alpha(t) \leq u'(t), \forall t \in [0, +\infty)$, and the remaining inequality $u(t) \leq \beta(t), \forall t \in [0, +\infty)$, can be proved using the same technique. \square

4 Example

Problems of boundary layer flow over a stretching sheet, with and without heat transfer, are a topic that arouses growing interest in the literature (see, for example [7, 15, 20, 23]). These papers deal with a boundary value problem of normal stagnation point flow impinging on a stretching sheet, governed by the parameter b which represents the ratio of the strain rate of the stagnation flow to that of the stretching sheet. Existing numerical studies on the basic flow shows that a solution exists for all values of $b > 0$.

In [22], the third order differential equation

$$f''' + f f'' - (f')^2 + b^2 = 0, \tag{4.1}$$

together with the boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = b,$$

is studied.

Motivated by this paper, in this application, we prove the solvability of the impulsive third order problem composed of a differential equation similar to

(4.1), namely,

$$u'''(t) = \begin{cases} -\frac{0.01}{1+t^2} \left(\sqrt[3]{u(t)}\sqrt{|u''(t)|} + \operatorname{sgn}(u'(t))\sqrt{|u'(t)|} - \operatorname{sgn}(u''(t))b^2 \right), & \text{if } 0 \leq t \leq 1, \\ -\frac{0.01}{1+t^2} \left(\sqrt[3]{u(t)}\sqrt{|u''(t)|} + \operatorname{sgn}(u'(t))\sqrt{|u'(t)|} + 10\operatorname{sgn}(u''(t))b^2 \right), & \text{if } t > 1, \end{cases} \quad (4.2)$$

with $b \in \mathbb{R} \setminus \{0\}$, where $u(t)$ represents the flow speed across a time t , together with the asymptotic boundary conditions

$$u(0) = A, \quad u'(0) = B, \quad u''(+\infty) = 0, \quad (4.3)$$

for $A, B \in \mathbb{R}$, and the impulsive effects with the form

$$\begin{aligned} \Delta u(t_k) &= \frac{1}{(t_k)^3} \left(\lambda_{01} \sqrt[3]{u(t_k)} + \lambda_{02} (u'(t_k)) + \lambda_{03} \sqrt[3]{u''(t_k)} \right), \\ \Delta u'(t_k) &= \frac{1}{(t_k)^3} \left(\lambda_{11} (u(t_k)) + \lambda_{12} \sqrt[3]{u'(t_k)} \right), \\ \Delta u''(t_k) &= \frac{1}{(t_k)^3} \left(\lambda_{21} \sqrt[3]{u(t_k)} + \lambda_{22} (u'(t_k)) + \lambda_{23} (u''(t_k)) + \operatorname{sgn}(u''(t_k)) * 5t_k \right), \end{aligned} \quad (4.4)$$

where $\lambda_{ij} \in \mathbb{R}$, for $1 = 0, 1, 2$ and $j = 1, 2, 3$, and $k \in \mathbb{N}$.

Note that:

1. The null function is not a solution of (4.2).
2. In (4.2), from a theoretical point of view, the parameter b could be non-positive.
3. For functions $u \in X$, the condition $u''(+\infty) = 0$ implies that $u'(+\infty)$ is finite.
4. Neither (4.2) nor (4.4) are covered by Theorem 3.1 of [29], as they are not sublinear and have different monotonicities.
5. The problem (4.2)–(4.4) is a particular case of the initial problem (1.1)–(1.3), with $C = 0$,

$$f(t, y_0, y_1, y_2) = \quad (4.5)$$

$$\begin{cases} -\frac{0.01}{1+t^2} \left(\sqrt[3]{y_0}\sqrt{|y_2|} + \operatorname{sgn}(y_1)\sqrt{|y_1|} - \operatorname{sgn}(y_2)b^2 \right), & \text{if } 0 \leq t \leq 1, \\ -\frac{0.01}{1+t^2} \left(\sqrt[3]{y_0}\sqrt{|y_2|} + \operatorname{sgn}(y_1)\sqrt{|y_1|} + 10 \operatorname{sgn}(y_2)b^2 \right), & \text{if } t > 1, \end{cases}$$

$$I_{0,1}(t_k, w_0, w_1, w_2) = \frac{1}{(t_k)^3} \left(\lambda_{01} \sqrt[3]{w_0} + \lambda_{02} (w_1) + \lambda_{03} \sqrt[3]{w_2} \right),$$

$$I_{1,1}(t_k, w_0, w_1, w_2) = \frac{1}{(t_k)^3} \left(\lambda_{11} (w_0) + \lambda_{12} \sqrt[3]{w_1} \right),$$

$$I_{2,1}(t_k, w_0, w_1, w_2) = \frac{1}{(t_k)^3} \left(\lambda_{21} \sqrt[3]{w_0} + \lambda_{22} (w_1) + \lambda_{23} (w_2) + \operatorname{sgn}(w_2)5t_k \right).$$

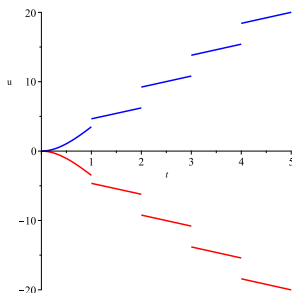


Figure 1. Solution region.

As a numeric example, let us consider $b = -1$, $A = B = 0$, $t_k = k$, $k \in \mathbb{N}$, and adequate values for the parameters.

In this case, the impulsive conditions are given by

$$\begin{aligned} \Delta u(k) &= \frac{1}{k^3} \left(0.001^k \sqrt[3]{u(k)} + 0.001^k (u'(k)) + 0.001^k \sqrt[3]{u''(k)} \right), \\ \Delta u'(k) &= \frac{1}{k^3} \left(0.1^k (u(k)) + 0.1^k \sqrt[3]{u'(k)} \right), \\ \Delta u''(k) &= \frac{1}{k^3} \left(-0.1^k \sqrt[3]{u(k)} - 0.1^k (u'(k)) + 0.1^k (u''(k)) + \text{sgn}(u''(k))5k \right), \end{aligned} \tag{4.6}$$

and the piecewise functions $\alpha, \beta \in X$ defined as

$$\begin{aligned} \alpha(t) &= \begin{cases} 1.5t^3 - 5t^2, & \text{if } 0 \leq t \leq 1, \\ -0.1 \left(\frac{1}{2t} + 16t \right) - 3k, & \text{if } t \in]k, k + 1], k \geq 1, \end{cases} \\ \beta(t) &= \begin{cases} -1.5t^3 + 5t^2, & \text{if } t \leq 1, \\ 0.1 \left(\frac{1}{2t} + 16t \right) + 3k, & \text{if } t \in]k, k + 1], k \geq 1, \end{cases} \end{aligned}$$

are, respectively, lower and upper solutions of problem (4.2), (4.3), (4.6), according to Definition 3, satisfying (3.1).

Moreover, the nonlinear part given by (4.5) verifies (3.2), the impulsive functions $I_{i,1} : \mathbb{R}^4 \mapsto \mathbb{R}$, $i = 0, 1, 2$,

$$\begin{aligned} 0, 1(k, w_0, w_1, w_2) &= \frac{1}{k^3} \left(0.001^k \sqrt[3]{w_0} + 0.001^k w_1 + 0.001^k \sqrt[3]{w_2} \right), \\ I_{1,1}(k, w_0, w_1, w_2) &= \frac{1}{k^3} \left(0.1^k w_0 + 0.1^k \sqrt[3]{w_1} \right), \\ I_{2,1}(k, w_0, w_1, w_2) &= \frac{1}{k^3} \left(-0.1^k \sqrt[3]{w_0} - 0.1^k w_1 + 0.1^k w_2 + \text{sgn}(w_2) * 5k \right), \end{aligned}$$

satisfy (3.3), (3.4) and (3.5), and assumption (A) holds for $\xi \geq 24.245$.

Then, by Theorem 2, there exists a solution $u \in X$ of problem (4.2), (4.3), (4.6), in the strip

$$\alpha(t) \leq u(t) \leq \beta(t), \text{ for } t \in (0, +\infty),$$

that is, in the region illustrated in Figure 1.

From the localization part of Theorem 2 we can also have some data on the first and second derivatives:

$$\alpha'(t) \leq u'(t) \leq \beta'(t), \quad \alpha''(t) \leq u''(t) \leq \beta''(t), \quad \text{for } t \in]0, +\infty),$$

that is the growth and concavity variations are in the strips given by Figure 2.

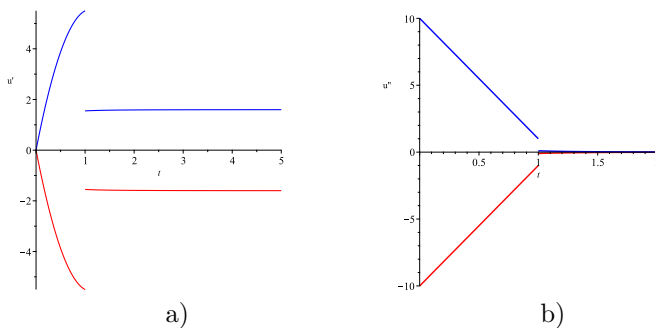


Figure 2. a) derivative region; b) concavity variation.

5 Conclusions

Higher-order boundary value problems on unbounded domains are more delicate, as the nonlinearities can be chaotic. From a theoretical point of view, the issue relies on the noncompactness of the associated operator, and to overcome it, some additional tools are required, such as the stability at ∞ . Moreover, in impulsive problems, with extra jumps, the stability on each impulsive moment must hold. Lower and upper solutions prove to be an adequate method and technique for these kinds of problems, as it gives not only the existence of a solution but also some qualitative data such as monotonicity, variations, concavity, . . . In this way, it enlarges the range of applications and it is very useful for nonlinear problems, where it is not possible to have an explicit solution.

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