Mathematical Modelling and Analysis
Volume 26, Issue 4, 519–527, 2021
https://doi.org/10.3846/mma.2021.13632

Uniform Regularity for the Isentropic Compressible Magneto-Micropolar System

Jishan Fan[a], Peng Wang[b] and Yong Zhou[b,c]

[a] *Department of Applied Mathematics, Nanjing Forestry University*
210037 Nanjing, China
[b] *School of Mathematics (Zhuhai), Sun Yat-sen University*
519082 Zhuhai, Guangdong, China
[c] *Department of Mathematics, Zhejiang Normal University*
321004 Jinhua, Zhejiang, China
E-mail(*corresp.*): zhouyong3@mail.sysu.edu.cn
E-mail: fanjishan@njfu.edu.cn
E-mail: wangp258@mail2.sysu.edu.cn

Received September 29, 2020; revised August 16, 2021; accepted August 17, 2021

Abstract. In this paper, we are concerned with the uniform regularity estimates of smooth solutions to the isentropic compressible magneto-micropolar system in $\mathbb{T}^3$. Under the assumption that $0 < \mu, \zeta, \tilde{\mu} < 1, 0 < \lambda + \mu, \tilde{\lambda} + \tilde{\mu} < 1, 0 < \frac{1}{C_0} \le \rho_0 \le C_0$, and by applying the classic bilinear commutator and product estimates, the uniform estimates of solutions to the isentropic compressible magneto-micropolar system are established in $H^s(\mathbb{T}^3)$ space, $s > \frac{5}{2}$.

Keywords: compressible, magneto-micropolar, uniform regularity.

AMS Subject Classification: 35Q35; 35M41; 35B40; 76A35; 76N15.

1 Introduction

In this paper, we consider the following isentropic compressible magneto-micropolar system [9]:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\begin{aligned}\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - (\mu+\zeta)\Delta u - (\lambda+\mu-\zeta)\nabla \operatorname{div} u \\ = 2\zeta \operatorname{rot} w + \operatorname{rot} b \times b,\end{aligned} \tag{1.2}$$

$$\partial_t(\rho w)+\mathrm{div}\,(\rho u\otimes w)-\tilde{\mu}\Delta w-(\tilde{\lambda}+\tilde{\mu})\nabla\mathrm{div}\,w+4\zeta w=2\zeta\mathrm{rot}\,u, \quad (1.3)$$

$$\partial_t b+\mathrm{rot}\,(b\times u)=\Delta b, \quad (1.4)$$

$$\mathrm{div}\,b=0 \quad \text{in} \quad \mathbb{T}^3\times(0,\infty), \quad (1.5)$$

$$(\rho,u,w,b)(\cdot,0)=(\rho_0,u_0,w_0,b_0)(\cdot) \quad \text{in} \quad \mathbb{T}^3. \quad (1.6)$$

Here the unknowns ρ, u, w and b stand for the density, velocity, micro-rational velocity, and magnetic field, respectively. The pressure $p := a\rho^\gamma$ with positive constants a and $\gamma > 1$. The parameters $\mu, \lambda, \zeta, \tilde{\mu}$ and $\tilde{\lambda}$ are constants denoting the viscosity coefficients satisfying

$$\mu,\zeta,\tilde{\mu}>0, 2\mu+3\lambda-4\zeta>0 \quad \text{and} \quad 2\tilde{\mu}+3\tilde{\lambda}\geq 0.$$

We have the well-known vector identities:

$$\mathrm{rot}\,(b\times u)=u\cdot\nabla b-b\cdot\nabla u+b\mathrm{div}\,u, \quad \mathrm{rot}\,b\times b=\mathrm{div}\,\left(b\otimes b-\frac{1}{2}|b|^2\mathbb{I}_3\right),$$

where the symbol $b\otimes b$ denotes a matrix whose (i,j)th entry is b_ib_j and $\mathbb{I}_3$ is the identity matrix of order 3.

We will assume the following natural compatibility conditions:

$$\nabla p(\rho_0)-(\mu+\zeta)\Delta u_0-(\lambda+\mu-\zeta)\nabla\mathrm{div}\,u_0-2\zeta\mathrm{rot}\,w_0-\mathrm{rot}\,b_0\times b_0=\sqrt{\rho_0}g_1, \quad (1.7)$$

$$-\tilde{\mu}\Delta w_0-(\tilde{\lambda}+\tilde{\mu})\nabla\mathrm{div}\,w_0+4\zeta w_0-2\zeta\mathrm{rot}\,u_0=\sqrt{\rho_0}g_2 \quad (1.8)$$

for some $(g_1, g_2) \in L^2$.

When $\zeta = 0$ and $w = 0$, (1.1), (1.2), (1.4) and (1.5) reduce to the isentropic compressible MHD system. Xu-Zhang [12] and Zhu-Chen [14] showed some regularity criteria with (1.7). Fan-Zhou [3] proved the local well-posedness of strong solutions without (1.7).

When $\zeta = 0$ and $b = 0$, (1.1) and (1.2) reduce to the isentropic compressible Navier-Stokes system. Gong-Li-Liu-Zhang [4] and Huang [5] proved the local well-posedness of strong solutions without (1.7).

Wei-Guo-Li [10] and Wu-Wang [11] studied the long-time behavior of smooth solutions when $\inf \rho_0 > 0$. Zhang [13] showed the local well-posedness (without proof) and a blow-up criterion with $\inf \rho_0 = 0$ and (1.7)–(1.8).

The aim of this paper is to prove uniform regularity estimates. We will prove

Theorem 1. *Let* $0 < \mu, \zeta, \tilde{\mu} < 1, 0 < \lambda + \mu, \tilde{\lambda} + \tilde{\mu} < 1, 0 < \frac{1}{C_0} \leq \rho_0 \leq C_0, \rho_0, u_0, w_0, b_0 \in H^s(\mathbb{T}^3)$ *with* $s > \frac{5}{2}$ *and* $\mathrm{div}\, b_0 = 0$ *in* $\mathbb{T}^3$. *Let* (ρ, u, w, b) *be the unique local smooth solutions to the problem* (1.1)–(1.6). *Then*

$$\|(\rho,u,w,b)(\cdot,t)\|_{H^s}\leq C \quad in \quad [0,T]$$

holds true for some positive constants C *and* T_0 $(\leq T)$ *independent of* $\lambda, \mu, \zeta, \tilde{\lambda}$ *and* $\tilde{\mu}$.

To prove Theorem 1, we will rewrite (1.1) as follows.

$$\frac{1}{\gamma p}\partial_t p + \frac{1}{\gamma p} u \cdot \nabla p + \operatorname{div} u = 0. \tag{1.9}$$

We define

$$\begin{aligned} M(t) : \quad = \quad & 1 + \sup_{0 \le s_1 \le t} \Big\{ \|(\rho, u, w, b, p)(\cdot, s_1)\|_{H^s} + \|\partial_t u(\cdot, s_1)\|_{L^2} \\ & + \|\partial_t w(\cdot, s_1)\|_{L^2} + \|1/\rho(\cdot, s_1)\|_{L^\infty} + \|1/p(\cdot, s_1)\|_{L^\infty} \Big\}. \end{aligned}$$

We can prove

Theorem 2. *For any $t \in [0, T_0)$ $(T_0 \le 1)$, we have that*

$$M(t) \le C_0(M_0) \exp(tC(M)) \tag{1.10}$$

for some nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

It follows from (1.10) that [1, 2, 7]:

$$M(t) \le C.$$

In the following proofs, we will use the bilinear commutator and product estimates due to Kato-Ponce [6]:

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{q_1}} + \|g\|_{L^{p_2}} \|\Lambda^s f\|_{L^{q_2}}), \tag{1.11}$$

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}) \tag{1.12}$$

with $s > 0, \Lambda := (-\Delta)^{\frac{1}{2}}$ and $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{p_2} + \dfrac{1}{q_2}$.

We only need to show Theorem 2.

2 Proof of Theorem 2

First, testing (1.1) by ρ^{q-1}, we see that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^q \mathrm{d}x = \left(1 - \frac{1}{q}\right)\int \rho^q \operatorname{div} u \mathrm{d}x \le \|\operatorname{div} u\|_{L^\infty} \int \rho^q \mathrm{d}x,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^q} \le \|\operatorname{div} u\|_{L^\infty} \|\rho\|_{L^q},$$

which gives

$$\|\rho\|_{L^q} \le \|\rho_0\|_{L^q} \exp\left(\int_0^t \|\operatorname{div} u\|_{L^\infty} \mathrm{d}\tau\right).$$

Taking $q \to +\infty$, we get

$$\|\rho\|_{L^\infty} \le \|\rho_0\|_{L^\infty} \exp(tC(M)). \tag{2.1}$$

It follows from (1.1) that

$$\partial_t \frac{1}{\rho} + u \cdot \nabla \frac{1}{\rho} - \frac{1}{\rho} \operatorname{div} u = 0. \tag{2.2}$$

Testing (2.2) by $(1/\rho)^{q-1}$, we find that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\frac{1}{\rho}\right)^q \mathrm{d}x = \left(1+\frac{1}{q}\right)\int \left(\frac{1}{\rho}\right)^q \operatorname{div} u \mathrm{d}x \le \left(1+\frac{1}{q}\right) \left\|\frac{1}{\rho}\right\|_{L^q}^q \|\operatorname{div} u\|_{L^\infty},$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{1}{\rho}\right\|_{L^q} \le \left(1+\frac{1}{q}\right)\left\|\frac{1}{\rho}\right\|_{L^q} \|\operatorname{div} u\|_{L^\infty},$$

which gives

$$\left\|\frac{1}{\rho}\right\|_{L^q} \le \left\|\frac{1}{\rho_0}\right\|_{L^q} \exp\left(\left(1+\frac{1}{q}\right)\int_0^t \|\operatorname{div} u\|_{L^\infty} \mathrm{d}\tau\right)$$

and we have

$$\left\|\frac{1}{\rho}\right\|_{L^\infty} \le \left\|\frac{1}{\rho_0}\right\|_{L^\infty} \exp(tC(M)) \tag{2.3}$$

by sending $q \to +\infty$. (2.1) and (2.3) give

$$\|p\|_{L^\infty} + \|1/p\|_{L^\infty} \le C_0(M_0)\exp(tC(M)). \tag{2.4}$$

It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2 \mathrm{d}x = 2\int u\partial_t u \mathrm{d}x \le 2\|u\|_{L^2}\|\partial_t u\|_{L^2} \le C(M),$$

which implies

$$\|u\|_{L^2} \le C_0(M_0)\exp(tC(M)). \tag{2.5}$$

Similarly, we have

$$\|w\|_{L^2} \le C_0(M_0)\exp(tC(M)). \tag{2.6}$$

Testing (1.4) by b, we derive

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b|^2\mathrm{d}x + \int |\nabla b|^2 \mathrm{d}x &= -\int (u\cdot\nabla b - b\cdot\nabla u + b\operatorname{div} u) b \mathrm{d}x \\ = -\int \left(\frac{1}{2}|b|^2 \operatorname{div} u - b\cdot\nabla u\cdot b\right)\mathrm{d}x &\le C\|\nabla u\|_{L^\infty}\|b\|_{L^2}^2 \le C(M),\end{aligned}$$

which leads to

$$\|b\|_{L^2}^2 + \int_0^t\int |\nabla b|^2 \mathrm{d}x\mathrm{d}\tau \le C_0(M_0)\exp(tC(M)). \tag{2.7}$$

Applying Λ^s to (1.9), testing by $\Lambda^s p$, using (1.9), (1.11) and (1.12), we compute

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{\gamma p}(\Lambda^s p)^2 \mathrm{d}x + \int \Lambda^s p \Lambda^s \operatorname{div} u \mathrm{d}x = \frac{1}{2}\int (\Lambda^s p)^2$$

$$
\begin{aligned}
&\times \left[\operatorname{div}\left(\frac{u}{\gamma p}\right) - \frac{1}{\gamma p^2}\partial_t p\right]\mathrm{d}x - \int \left(\Lambda^s\left(\frac{1}{\gamma p}\partial_t p\right) - \frac{1}{\gamma p}\Lambda^s\partial_t p\right)\Lambda^s p \mathrm{d}x \\
&- \int \left(\Lambda^s\left(\frac{u}{\gamma p}\cdot\nabla p\right) - \frac{u}{\gamma p}\cdot\nabla\Lambda^s p\right)\Lambda^s p\mathrm{d}x \le C\|\Lambda^s p\|_{L^2}^2 \\
&\times \left\|\operatorname{div}\left(\frac{u}{\gamma p}\right) - \frac{1}{\gamma p^2}\partial_t p\right\|_{L^\infty} + C\|\partial_t p\|_{L^\infty}\left\|\Lambda^s\left(\frac{1}{\gamma p}\right)\right\|_{L^2}\|\Lambda^s p\|_{L^2} \\
&+ C\left\|\nabla\frac{1}{\gamma p}\right\|_{L^\infty}\|\Lambda^{s-1}\partial_t p\|_{L^2}\|\Lambda^s p\|_{L^2} + C\|\nabla p\|_{L^\infty}\left\|\Lambda^s\left(\frac{u}{\gamma p}\right)\right\|_{L^2}\|\Lambda^s p\|_{L^2} \\
&+ C\left\|\nabla\frac{u}{\gamma p}\right\|_{L^\infty}\|\Lambda^s p\|_{L^2}^2 \le C(M) + C(M)\|\partial_t p\|_{L^\infty} + C(M)\|\Lambda^{s-1}\partial_t p\|_{L^2} \\
&\le C(M) + C(M)\|u\cdot\nabla p + \gamma p\operatorname{div} u\|_{L^\infty} \\
&+ C(M)\|\Lambda^{s-1}(u\cdot\nabla p + \gamma p\operatorname{div} u)\|_{L^2} \le C(M). \qquad (2.8)
\end{aligned}
$$

Here we have used the estimate [8]:

$$\|\Lambda^s 1/p\|_{L^2} \le C(M)\|\Lambda^s p\|_{L^2} \le C(M).$$

It is obvious that

$$\int_0^t \int |\partial_t u|^2 \mathrm{d}x\mathrm{d}\tau \le t\sup\int |\partial_t u|^2\mathrm{d}x \le tC(M). \qquad (2.9)$$

Applying Λ^{s-1} to (1.2), testing by $\Lambda^{s-1}\partial_t u$, using (1.11) and (1.12), we obtain

$$
\begin{aligned}
&\frac{\mu+\zeta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\Lambda^s u|^2\mathrm{d}x + \frac{\lambda+\mu-\zeta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\Lambda^{s-1}\operatorname{div} u)^2\mathrm{d}x + \int \rho|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x \\
&= 2\zeta\int \Lambda^{s-1}\operatorname{rot} w\Lambda^{s-1}\partial_t u\mathrm{d}x + \int \Lambda^{s-1}\left((b\cdot\nabla b) - \frac{1}{2}\nabla|b|^2\right)\Lambda^{s-1}\partial_t u\mathrm{d}x \\
&\quad - \int \Lambda^{s-1}\nabla p\cdot\Lambda^{s-1}\partial_t u\mathrm{d}x - \int \Lambda^{s-1}(\rho u\cdot\nabla u)\cdot\Lambda^{s-1}\partial_t u\mathrm{d}x \\
&\quad - \int [\Lambda^{s-1}(\rho\partial_t u) - \rho\Lambda^{s-1}\partial_t u]\Lambda^{s-1}\partial_t u\mathrm{d}x \le C\|w\|_{H^s}\|\Lambda^{s-1}\partial_t u\|_{L^2} \\
&\quad + C\|b\|_{H^s}^2\|\Lambda^{s-1}\partial_t u\|_{L^2} + C\|\rho\|_{H^{s-1}}\|u\|_{H^s}^2\|\Lambda^{s-1}\partial_t u\|_{L^2} \\
&\quad + C\|\Lambda^s p\|_{L^2}\|\Lambda^{s-1}\partial_t u\|_{L^2} + \|\partial_t u\|_{L^\infty}\|\Lambda^{s-1}\rho\|_{L^2})\|\Lambda^{s-1}\partial_t u\|_{L^2} \\
&\quad + C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-2}\partial_t u\|_{L^2} \\
&\le C(M)\|\Lambda^{s-1}\partial_t u\|_{L^2} + C(M)(\|\Lambda^{s-2}\partial_t u\|_{L^2} + \|\partial_t u\|_{L^\infty})\|\Lambda^{s-1}\partial_t u\|_{L^2} \\
&\le C(M)\|\Lambda^{s-1}\partial_t u\|_{L^2} + C(M)\left(\|\partial_t u\|_{L^2}^{\frac{1}{s-1}}\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{s-2}{s-1}} + \|\partial_t u\|_{L^2}\right. \\
&\quad \left. + \|\partial_t u\|_{L^2}^{\frac{s-1-\frac{n}{2}}{s-1}}\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{n}{2(s-1)}}\right)\|\Lambda^{s-1}\partial_t u\|_{L^2} \\
&\le C(M)\|\Lambda^{s-1}\partial_t u\|_{L^2} + C(M)(\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{s-2}{s-1}} \\
&\quad + \|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{n}{2(s-1)}})\|\Lambda^{s-1}\partial_t u\|_{L^2} \le \frac{1}{2}\int \rho|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x + C(M),
\end{aligned}
$$

which gives

$$\int_0^t \int |\Lambda^{s-1}\partial_t u|^2 \mathrm{d}x\mathrm{d}\tau \le C_0(M_0)\exp(tC(M)). \tag{2.10}$$

Similarly to (2.9), we infer that

$$\int_0^t \int |\partial_t w|^2 \mathrm{d}x\mathrm{d}\tau \le tC(M).$$

Similarly to (2.10), applying Λ^{s-1} to (1.3), testing by $\Lambda^{s-1}\partial_t w$, using (1.11) and (1.12), we have

$$\begin{aligned}
&\frac{\tilde\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\Lambda^s w|^2 \mathrm{d}x + \frac{\tilde\lambda+\tilde\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\Lambda^{s-1}\mathrm{div}\, w)^2\mathrm{d}x + \int \rho|\Lambda^{s-1}\partial_t w|^2\mathrm{d}x \\
&\quad + 2\zeta\frac{\mathrm{d}}{\mathrm{d}t}\int |\Lambda^{s-1}w|^2\mathrm{d}x = 2\zeta\int \Lambda^{s-1}\mathrm{rot}\, u\cdot\Lambda^{s-1}\partial_t w\mathrm{d}x \\
&- \int \Lambda^{s-1}(\rho u\cdot\nabla w)\Lambda^{s-1}\partial_t w\mathrm{d}x - \int [\Lambda^{s-1}(\rho\partial_t w) - \rho\Lambda^{s-1}\partial_t w]\Lambda^{s-1}\partial_t w\mathrm{d}x \\
&\le C\|\Lambda^s u\|_{L^2}\|\Lambda^{s-1}\partial_t w\|_{L^2} + C(\|\rho u\|_{L^\infty}\|\Lambda^s w\|_{L^2} + \|\nabla w\|_{L^\infty}\|\Lambda^{s-1}(\rho u)\|_{L^2}) \\
&\times \|\Lambda^{s-1}\partial_t w\|_{L^2} + C\left(\|\partial_t w\|_{L^\infty}\|\Lambda^{s-1}\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|\Lambda^{s-2}\partial_t w\|_{L^2}\right) \\
&\quad \times \|\Lambda^{s-1}\partial_t w\|_{L^2} \le C(M)\|\Lambda^{s-1}\partial_t w\|_{L^2} + C(M)\left(\|\partial_t w\|_{L^\infty} + \|\Lambda^{s-2}\partial_t w\|_{L^2}\right) \\
&\quad \times \|\Lambda^{s-1}\partial_t w\|_{L^2} \le C(M)\|\Lambda^{s-1}\partial_t w\|_{L^2} + C(M) \\
&\quad \times \left(\|\partial_t w\|_{L^2}^{\frac{s-\frac52}{s-1}}\|\Lambda^{s-1}\partial_t w\|_{L^2}^{\frac{3}{2(s-1)}} + \|\partial_t w\|_{L^2}^{\frac{1}{s-1}}\|\Lambda^{s-1}\partial_t w\|_{L^2}^{\frac{s-2}{s-1}}\right)\|\Lambda^{s-1}\partial_t w\|_{L^2} \\
&\le C(M)\|\Lambda^{s-1}\partial_t w\|_{L^2} + C(M)\left(\|\Lambda^{s-1}\partial_t w\|_{L^2}^{\frac{3}{2(s-1)}} + \|\Lambda^{s-1}\partial_t w\|_{L^2}^{\frac{s-2}{s-1}}\right) \\
&\quad \times \|\Lambda^{s-1}\partial_t w\|_{L^2} \le \frac12\int \rho|\Lambda^{s-1}\partial_t w|^2\mathrm{d}x + C(M),
\end{aligned}$$

which implies

$$\int_0^t \int |\Lambda^{s-1}\partial_t w|^2 \mathrm{d}x\mathrm{d}\tau \le C_0(M_0)\exp(tC(M)). \tag{2.11}$$

Applying Λ^s to (1.2), testing by $\Lambda^s u$, using (1.1), (1.11) and (1.12), we have

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\Lambda^s u|^2\mathrm{d}x + (\mu+\zeta)\int |\Lambda^{s+1}u|^2\mathrm{d}x + (\lambda+\mu-\zeta)\int (\Lambda^s\mathrm{div}\, u)^2\mathrm{d}x \\
&\quad + \int \Lambda^s\nabla p\cdot\Lambda^s u\mathrm{d}x = \int (\Lambda^s(b\cdot\nabla b) - b\cdot\nabla\Lambda^s b)\Lambda^s u\mathrm{d}x \\
&\quad - 2\zeta\int \Lambda^s w\cdot\Lambda^s\mathrm{rot}\, u\mathrm{d}x + \int b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\mathrm{d}x - \int \frac12\Lambda^s\nabla|b|^2\cdot\Lambda^s u\mathrm{d}x \\
&\quad - \int (\Lambda^s(\rho\partial_t u) - \rho\Lambda^s\partial_t u)\Lambda^s u\mathrm{d}x - \int (\Lambda^s(\rho u\cdot\nabla u) - \rho u\cdot\nabla\Lambda^s u)\Lambda^s u\mathrm{d}x
\end{aligned}$$

$$\leq C\zeta\|\Lambda^s w\|_{L^2}\|\Lambda^{s+1}u\|_{L^2} + C\|\nabla b\|_{L^\infty}\|\Lambda^s b\|_{L^2}\|\Lambda^s u\|_{L^2} + \int b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\mathrm{d}x$$
$$+ C\|b\|_{L^\infty}\|\Lambda^{s+1}b\|_{L^2}\|\Lambda^s u\|_{L^2} + C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-1}\partial_t u\|_{L^2} + \|\partial_t u\|_{L^\infty}\|\Lambda^s\rho\|_{L^2})$$
$$\times\|\Lambda^s u\|_{L^2} + C(\|\nabla u\|_{L^\infty}\|\Lambda^s(\rho u)\|_{L^2} + \|\nabla(\rho u)\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s u\|_{L^2}$$
$$\leq C(M)\zeta\|\Lambda^{s+1}u\|_{L^2} + C(M) + \int b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\mathrm{d}x + C(M)\|\Lambda^{s+1}b\|_{L^2}$$
$$+ C(M)(\|\Lambda^{s-1}\partial_t u\|_{L^2} + \|\partial_t u\|_{L^\infty}) \leq C(M) + \int b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\mathrm{d}x$$
$$+ \frac{1}{16}\|\Lambda^{s+1}b\|_{L^2}^2 + \|\Lambda^{s-1}\partial_t u\|_{L^2}^2 + \frac{\zeta}{8}\|\Lambda^{s+1}u\|_{L^2}^2. \tag{2.12}$$

Applying Λ^s to (1.3), testing by $\Lambda^s w$, using (1.1), (1.11) and (1.12), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\Lambda^s w|^2\mathrm{d}x + \tilde{\mu}\int|\Lambda^{s+1}w|^2\mathrm{d}x + (\tilde{\lambda}+\tilde{\mu})\int(\Lambda^s\mathrm{div}\, w)^2\mathrm{d}$$
$$+ 4\zeta\int|\Lambda^s w|^2\mathrm{d}x = 2\zeta\int\Lambda^s\mathrm{rot}\, u\cdot\Lambda^s w\mathrm{d}x - \int(\Lambda^s(\rho\partial_t w) - \rho\Lambda^s\partial_t w)\Lambda^s w\mathrm{d}x$$
$$- \int(\Lambda^s(\rho u\cdot\nabla w) - \rho u\cdot\nabla\Lambda^s w)\Lambda^s w\mathrm{d}x \leq C\zeta\|\Lambda^{s+1}u\|_{L^2}\|\Lambda^s w\|_{L^2}$$
$$+ C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-1}\partial_t w\|_{L^2} + \|\partial_t w\|_{L^\infty}\|\Lambda^s\rho\|_{L^2})\|\Lambda^s w\|_{L^2}$$
$$+ C(\|\nabla w\|_{L^\infty}\|\Lambda^s(\rho u)\|_{L^2} + \|\nabla(\rho u)\|_{L^\infty}\|\Lambda^s w\|_{L^2})\|\Lambda^s w\|_{L^2}$$
$$\leq C(M)\zeta\|\Lambda^{s+1}u\|_{L^2} + C(M)(\|\Lambda^{s-1}\partial_t w\|_{L^2} + \|\partial_t w\|_{L^\infty}) + C(M)$$
$$\leq C(M) + \frac{\zeta}{8}\|\Lambda^{s+1}u\|_{L^2}^2 + \|\Lambda^{s-1}\partial_t w\|_{L^2}^2. \tag{2.13}$$

Applying Λ^s to (1.4), testing by $\Lambda^s b$, using (1.11) and (1.12), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\Lambda^s b|^2\mathrm{d}x + \int|\Lambda^{s+1}b|^2\mathrm{d}x = -\int(\Lambda^s(u\cdot\nabla b) - u\cdot\nabla\Lambda^s b)\Lambda^s b\mathrm{d}x$$
$$- \int u\cdot\nabla\Lambda^s b\cdot\Lambda^s b\mathrm{d}x + \int(\Lambda^s(b\cdot\nabla u) - b\cdot\nabla\Lambda^s u)\Lambda^s b\mathrm{d}x + \int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\mathrm{d}x$$
$$- \int(\Lambda^s(b\mathrm{div}\, u) - b\Lambda^s\mathrm{div}\, u)\Lambda^s b\mathrm{d}x - \int b\Lambda^s\mathrm{div}\, u\Lambda^s b\mathrm{d}x$$
$$\leq C(\|\nabla u\|_{L^\infty}\|\Lambda^s b\|_{L^2} + \|\nabla b\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s b\|_{L^2}$$
$$+ \int\frac{1}{2}|\Lambda^s b|^2\mathrm{div}\, u\mathrm{d}x + \int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\mathrm{d}x + \int\Lambda^s u\cdot\nabla(b\Lambda^s b)\mathrm{d}x$$
$$\leq C(M) + \int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\mathrm{d}x + C\|\Lambda^s u\|_{L^2}(\|b\|_{L^\infty}\|\Lambda^{s+1}b\|_{L^2}$$
$$+ \|\nabla b\|_{L^\infty}\|\Lambda^s b\|_{L^2}) \leq C(M) + \int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\mathrm{d}x + C(M)\|\Lambda^{s+1}b\|_{L^2}$$
$$\leq \frac{1}{16}\|\Lambda^{s+1}b\|_{L^2}^2 + C(M) + \int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\mathrm{d}x. \tag{2.14}$$

Summing up (2.8), (2.12), (2.13) and (2.14), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{\gamma p}(\Lambda^s p)^2\mathrm{d}x + \rho|\Lambda^s u|^2 + \rho|\Lambda^s w|^2 + |\Lambda^s b|^2\right)\mathrm{d}x + \mu\int|\Lambda^{s+1}u|^2\mathrm{d}x$$

$$
\begin{aligned}
&+(\lambda+\mu-\zeta)\int(\Lambda^s\mathrm{div}\,u)^2\mathrm{d}x+\frac{7}{8}\int|\Lambda^{s+1}b|^2\mathrm{d}x+\frac{3}{4}\zeta\int|\Lambda^{s+1}u|^2\mathrm{d}x\\
&+\tilde{\mu}\int|\Lambda^{s+1}w|^2\mathrm{d}x+(\tilde{\lambda}+\tilde{\mu})\int(\Lambda^s\mathrm{div}\,w)^2\mathrm{d}x+4\zeta\int|\Lambda^s w|^2\mathrm{d}x\\
&+\int(\Lambda^s p\Lambda^s\mathrm{div}\,u+\Lambda^s\nabla p\cdot\Lambda^s u)\mathrm{d}x\\
\leq&C(M)+\|\Lambda^{s-1}\partial_t u\|_{L^2}^2+\|\Lambda^{s-1}\partial_t w\|_{L^2}^2+\int(b\cdot\nabla)(\Lambda^s b\cdot\Lambda^s u)\mathrm{d}x. \qquad (2.15)
\end{aligned}
$$

Notice that the last term of the LHS and the last term of RHS in (2.15) are zeros, using (2.10) and (2.11), we have

$$
\|\Lambda^s(p,u,w,b)\|_{L^2}\leq C_0(M_0)\exp(tC(M)). \qquad (2.16)
$$

On the other hand, it follows from (1.2) that

$$
\begin{aligned}
\|\partial_t u\|_{L^2}=&\Big\|\frac{1}{\rho}\Big(2\zeta\mathrm{rot}\,w+b\cdot\nabla b-\frac{1}{2}\nabla|b|^2+(\mu+\zeta)\Delta u+(\lambda+\mu-\zeta)\nabla\mathrm{div}\,u\\
&-\nabla p-\rho u\cdot\nabla u\Big)\Big\|_{L^2}\leq C_0(M_0)\exp(tC(M)). \qquad (2.17)
\end{aligned}
$$

Using the following estimate [8]:

$$
\|\Lambda^s\rho\|_{L^2}\leq C(1+\|p\|_{L^\infty})^\sigma\|f'\|_{W^{\sigma,\infty}(I)}\|\Lambda^s p\|_{L^2}
$$

with $\rho=f(p):=\left(\frac{p}{a}\right)^{\frac{1}{\gamma}}$, and

$$
I\subset\left(\frac{1}{C_0(M_0)}\exp(-tC(M)),C_0(M_0)\exp(tC(M))\right),
$$

and σ is an integer satisfying $\sigma\geq s$. We have

$$
\|\Lambda^s\rho\|_{L^2}\leq C_0(M_0)\exp(tC(M)). \qquad (2.18)
$$

Combining (2.3)–(2.7), (2.16), (2.17), and (2.18), we conclude that (1.10) holds true. This completes the proof.

Acknowledgements

This work is partially supported by NSFC (No. 11191234).

References

[1] T. Alazard. Low Mach number limit of the full Navier-Stokes equations. *Archive for Rational Mechanics and Analysis*, **180**(1):1–73, 2006. https://doi.org/10.1007/s00205-005-0393-2.

[2] C. Dou, S. Jiang and Y. Ou. Low Mach number limit of full Navier-Stokes equations in a 3D bounded domain. *Journal of Differential Equations*, **258**(2):379–398, 2015. https://doi.org/10.1016/j.jde.2014.09.017.

[3] J. Fan and Y. Zhou. Local well-posedness for the isentropic compressible MHD system with vacuum. *Journal of Mathematical Physics*, **62**(5):051505, 2021. https://doi.org/10.1063/5.0029046.

[4] H. Gong, J. Li, X.-G. Liu and X. Zhang. Local well-posedness of isentropic compressible Navier-Stokes equations with vacuum. *Communications in Mathematical Sciences*, **18**(7):1891–1909, 2020. https://doi.org/10.4310/CMS.2020.v18.n7.a4.

[5] X. Huang. On local strong and classical solutions to the three-dimensional barotropic compressible Navier-Stokes equations with vacuum. *Science China Mathematics*, 2020. https://doi.org/10.1007/s11425-019-9755-3.

[6] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Communications Pure and Applied Mathematics*, **41**(7):891–907, 1988. https://doi.org/10.1002/cpa.3160410704.

[7] G. Métivier and S. Schochet. The incompressible limit of the non-isentropic Euler equations. *Archive for Rational Mechanics and Analysis*, **158**(1):61–90, 2001. https://doi.org/10.1007/PL00004241.

[8] H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Springer, Basel, 1983. https://doi.org/10.1007/978-3-0346-0416-1.

[9] G. Łukaszewicz. *Micropolar Fluids. Modeling and Simulation in Science, Engineering and Technology.* Birkhäuser Boston, Inc., Boston, MA, 1999. https://doi.org/10.1007/978-1-4612-0641-5_5.

[10] R. Wei, B. Guo and Y. Li. Global existence and optimal convergence rates of solutions for 3D compressible magneto-micropolar fluid equations. *Journal of Differential Equations*, **263**(5):2457–2480, 2017. https://doi.org/10.1016/j.jde.2017.04.002.

[11] Z. Wu and W. Wang. The pointwise estimates of diffusion wave of the compressible micropolar fluids. *Journal of Differential Equations*, **265**(6):2544–2576, 2018. https://doi.org/10.1016/j.jde.2018.04.039.

[12] X. Xu and J. Zhang. A blow-up criterion for 3D compressible magnetohydrodynamic equations with vacuum. *Mathematical Models and Methods in Applied Sciences*, **22**(2):1150010, 2012. https://doi.org/10.1142/S0218202511500102.

[13] P. Zhang. Blow-up criterion for 3D compressible viscous magneto-micropolar fluids with initial vacuum. *Boundary Value Problems*, **2013**(160):1–16, 2013. https://doi.org/10.1186/1687-2770-2013-160.

[14] L. Zhu and Y. Chen. A new blowup criterion for strong solutions to the Cauchy problem of three-dimensional compressible magnetohydrodynamic equations. *Nonlinear Analysis: Real World Applications*, **41**:461–474, 2018. https://doi.org/10.1016/j.nonrwa.2017.10.018.