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A Numerical Method for Solving a Complete Hypersingular Integral Equation of the Second Kind and Its Justification

Oleksii V. Kostenko

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine
47 Nauky Ave., 61103 Kharkiv, Ukraine
E-mail(*corresp.*): alexvladkost@gmail.com; kostenko@ilt.kharkov.ua

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Abstract. A complete hypersingular integral equation of the second kind was obtained as a boundary integral equation for the diffraction and scattering problem of electromagnetic waves in space separated by the periodically placed non-perfectly conducting strips. The equation includes a singular integral that distinguishes it from the studied second-kind hypersingular equation. Our motivation is the need to have a numerical method for the equation, its applicability borders, and guaranteed convergence. The numerical method has the type of Nyström. The justification of the method envelops a proof of the theorem of existence and uniqueness of the solution and an estimate of the convergence rate of sequence of the approximate solutions to an exact solution.

Keywords: complete hypersingular integral equation, singular integral, integral with the logarithmic kernel, numerical method, Nyström-type, existence, uniqueness, convergence rate, model problem, numerical convergence.

AMS Subject Classification: 65R20; 78M25; 65N22.

1 Introduction

There is a wide set of important applied and theoretical problems of electromagnetic wave propagation in space separated by a periodic strip lattice. Particularly and just for in-house examples some of them are shown in [10,31]. The theoretical description for such problems is collected in [10] and some applied results for resonances are enveloped by [31]. One approach to solving such problems is to reduce the original boundary value problem for the system of Maxwell's equations to boundary integral equations and their numerical analysis. The main advantage of this analytical and numerical approach is the transformation of the problem in the infinite domain to the problem in the finite one. That is a very important point for numerical analysis. The analytical transformation provides us an opportunity to take into account the subtle effects of medium-frequency wave processes. Three types of the different-kind boundary integral equations are obtained using this approach. These are a hypersingular equation, a singular equation, and an equation with a logarithmic kernel.

The complete hypersingular integral equation includes all special singular character integrals. These are the hypersingular integral, the singular integral, and the logarithmic kernel integral. In [17] and [19] we obtained this equation when constructing a mathematical model of diffraction and scattering of waves in space separated by a periodic lattice of non-perfectly conducting impedance strips. Let us note that the lattice can be supplemented with an active resonant layer. By the way, the form of the hypersingular equation will be preserved. A pair of the lattice and layer is less studied and attracts attention as shown for example in [30]. Also, the application area of the hypersingular integral equation is well-known to include the problems of acoustics, the extra attractive problems of the mathematical theory of the laser, and the problems of description and analysis of the metamaterials. As an example, a related laser theory problem with a hypersingular equation is shown in [2]. In [32], a metamaterial surface is described. A full frequency analysis of its electrodynamics in the terahertz range is presented. The analysis is based on a hypersingular integral equation. The subtle effects of the interaction of the lattice and waves are caught and described.

The paper is about the complete hypersingular integral equation of the second kind, the numerical method for solving this hypersingular equation, and the justification of the method. The justification includes the criterion for the existence and uniqueness of a solution of the hypersingular equation and the estimation of the convergence rate of the numerical method. It is about estimating a norm of the difference between the approximate and exact solutions. The presentation of these results is the aim of this paper. Also, the aim of the paper is the demonstrate and analyze the numerical convergence of the method using the model problem. The model problem is the special case of the complete hypersingular integral equation of the second kind with its known exact solution. Based on the results for the model problem, we characterized the numerical convergence of the method.

The presented results continue, develop, and are based on the series of works [11,12,16,18] devoted to the qualitative theory of hypersingular integral equations and numerical methods. Another standing shoulder for the presented results is the book [22].

The paper consists of ten sections and a list of references. The opening and final sections have a traditional and natural character. In Section 2, we state the problem and define the hypersingular and singular integrals. Section 3 includes the definitions of polynomial Hilbert spaces, their completions in the norms, linear operators generated by the hypersingular integral equation and its properties. Sections 4 and 5 contain the results on regularization of operators and their polynomial discretization by Nyström way and using the quadrature formulae. These are the main results of the numerical method for solving the hypersingular equation. Section 6 is about the questions of the existence and uniqueness of a solution. The section contains the main result on the justification of the numerical method. This is a criterion for the existence and uniqueness of a solution. In Section 7 we have the second part of the justification of the numerical method. It is the estimation of the rate of convergence of a sequence of approximate solutions to an exact one. Section 8 presents the model problem and analysis of numerical convergence. The final section summarizes the presented results and concludes the paper.

2 Problem statement

The complete hypersingular integral equation of the second kind has the following form:

$$\begin{aligned} &hu\left(y\right)\sqrt{1-y^2}-\frac{1}{\pi}\int_{-1}^{1}\frac{u\left(t\right)}{\left(y-t\right)^2}\sqrt{1-t^2}dt\\ &+\frac{a}{\pi}\int_{-1}^{1}\frac{u\left(t\right)}{t-y}\sqrt{1-t^2}dt+\frac{b}{\pi}\int -1^1\ln\left|t-y\right|u\left(t\right)\sqrt{1-t^2}dt\\ &+\frac{1}{\pi}\int_{-1}^{1}K\left(t,y\right)u\left(t\right)\sqrt{1-t^2}dt=f\left(y\right), \end{aligned} \tag{2.1}$$

where the variable y belongs to the interval $(-1,1)$. The parameters h, a, and b are given non-zero arbitrary constants. These numbers can be complex.

In all terms on the left-hand side of Equation (2.1), the unknown function u is multiplied by the square root function. At a first glance, this point complicates the form of the equation. Intuitively, it is natural to introduce a new unknown function and simplify the equation. But this is not rational. Firstly, similar equations are connected with the set of important and attractive applied problems as shown in [2,10,11,17,19,22,24,30,31,32] and discussed above. The square root factor corresponds to a condition of the boundary value problems. Secondly, in the general case, the functions u and $\sqrt{1-t^2}$ have different smoothness. Thirdly, the function $\sqrt{1-t^2}$ is the orthogonality weight of Chebyshev polynomials of the second kind. This is a good basis for choosing and using these polynomials in numerical analysis. But let us note that there are many important applied problems connected with the hypersingular integral equations without the square root function. Moreover, for example in [1,5,28], similar hypersingular equations are investigated. We also want to note that the terms from the left-hand side of Equation (2.1) have trigonometric analogs without the square root function. These are

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{u\left(\varphi\right)}{2\sin^2\frac{\theta-\varphi}{2}}d\varphi,\ \frac{1}{2\pi}\int_0^{2\pi}\cot\left(\frac{\theta-\varphi}{2}\right)u\left(\varphi\right)d\varphi,\ \frac{1}{2\pi}\int_0^{2\pi}\ln\left|\sin\frac{\theta-\varphi}{2}\right|u\left(\varphi\right)d\varphi.$$

Such hypersingular integral equations are a separate branch of the theory but are qualitatively close to Equation (2.1).

The second term on the left-hand side of Equation (2.1) is called the hypersingular integral. It is understood in the following sense: for any smooth function u that has α-Hölder continuous first derivative we have

$$\int_{-1}^{1} \frac{u\left(t\right)}{\left(t-y\right)^{2}}\sqrt{1-t^{2}}dt$$
$$= \lim_{\varepsilon\to 0}\Big(\int_{-1}^{y-\varepsilon} \frac{u\left(t\right)}{\left(t-y\right)^{2}}\sqrt{1-t^{2}}dt + \int_{y+\varepsilon}^{1} \frac{u\left(t\right)}{\left(t-y\right)^{2}}\sqrt{1-t^{2}}dt - 2\frac{u\left(y\right)}{\varepsilon}\sqrt{1-y^{2}}\Big).$$

The third term on the left-hand side of Equation (2.1) is called the singular integral. It is understood in the following sense: for any α-Hölder continuous function u we have

$$\int_{-1}^{1} \frac{u\left(t\right)}{y-t}\sqrt{1-t^{2}}dt = \lim_{\varepsilon\to 0}\Big(\int_{-1}^{y-\varepsilon} \frac{u\left(t\right)}{y-t}\sqrt{1-t^{2}}dt + \int_{y+\varepsilon}^{1} \frac{u\left(t\right)}{y-t}\sqrt{1-t^{2}}dt\Big).$$

The fourth term on the left-hand side of Equation (2.1) is naturally called the logarithmic kernel integral. It is understood as an improper integral.

We say that Equation (2.1) is called complete since it contains all special singular character integrals mentioned above. Let $C_{[-1,1]}^{r,\alpha}$ be the set of continuous on the segment $[-1,1]$ functions having all derivatives up to r-order and r-derivative is α-Hölder continuous. Suppose the function f belongs to the set $C_{[-1,1]}^{0,\alpha}$, the two-variables function K belongs to the set $C_{[-1,1]}^{1,\alpha}$ in each variable uniformly with respect to other one, and unknown function u belongs to the set $C_{[-1,1]}^{1,\alpha}$. So, the function u belonging to the set $C_{[-1,1]}^{0,\alpha}$ is the sufficient condition for the existence of the singular integral. Let us show it. We have that

$$\int_{-1}^{y-\varepsilon} \frac{u\left(t\right)}{t-y}\sqrt{1-t^{2}}dt + \int_{y+\varepsilon}^{1} \frac{u\left(t\right)}{t-y}\sqrt{1-t^{2}}dt = \int_{-1}^{l} \frac{u\left(t\right)-u\left(y\right)}{t-y}\sqrt{1-t^{2}}dt$$
$$+\int_{l}^{y-\varepsilon} \frac{u\left(t\right)-u\left(y\right)}{t-y}\sqrt{1-t^{2}}dt + \int_{y+\varepsilon}^{r} \frac{u\left(t\right)-u\left(y\right)}{t-y}\sqrt{1-t^{2}}dt + \int_{r}^{1} \frac{u\left(t\right)-u\left(y\right)}{t-y}$$
$$\times\sqrt{1-t^{2}}dt + u\left(y\right)\int_{-1}^{y-\varepsilon} \frac{1}{t-y}\sqrt{1-t^{2}}dt + u\left(y\right)\int_{y+\varepsilon}^{1} \frac{1}{t-y}\sqrt{1-t^{2}}dt.$$

So, there are six integrals on the right-hand side. The first and fourth exist as improper integrals. The second and third exist too. The function $\sqrt{1-t^2}$ is differentiable on the segment $[l,r]$. Since the function u is α-Hölder continuous, we have a constant C that the function $\frac{u(t)-u(y)}{t-y}$ is majorized by the integrable function $\frac{C}{|t-y|^{1-\alpha}}$ and is integrable. The sum of the fifth and sixth integrals is equal to $\pi y u\left(y\right)$ when the number ε tends to 0. Finally, this means that

$$\int_{-1}^{1} \frac{u\left(t\right)}{t-y}\sqrt{1-t^{2}}dt = \int_{-1}^{1} \frac{u\left(t\right)-u\left(y\right)}{t-y}\sqrt{1-t^{2}}dt + \pi y u\left(y\right).$$

In the same way, the function u belonging to the set $C_{[-1,1]}^{1,\alpha}$ is the sufficient condition for the existence of the hypersingular integral.

The parameters h, a, and b have an added physical sense for the diffraction and scattering problems as presented in [17,19,32]. The parameters depend on a strip width and an incident wave number. Also, the parameter h characterizes conductivity. If $|h|$ is equal to 0, then it corresponds to the perfect conductivity. The parameter a depends on an angle between a wave vector of the incident electromagnetic field and the normal vector of the lattice plane. If $|a|$ is equal to 0, then it corresponds to the orthogonal direction of the incident field. If the parameter b is equal to 0, then it corresponds to the equation of the wing aerodynamics problem. Let us note again that some problems of aerodynamics can be reduced to the hypersingular equation of form (2.1). These parameters and the corresponding terms in the hypersingular equation appeared as the form and content of strip lattice became more complicated.

Some special cases of Equation (2.1) have already been studied. The case $|h|$ is equal to 0 was considered by Prof. Yuriy Gandel and Oleksii Kononenko in [12]. The case $|a|$ is equal to 0 was analyzed in our paper [16]. Thus, this paper has a base, expands, and enriches theory of hypersingular integral equations. Also, the presented results increase the scope of the diffraction and scattering theory.

Therefore the integral equations of form (2.1) are helping to solve the applied problems of mathematics. As mentioned above, in particular, similar equations can be successfully applied to the mathematical modelling of wave processes. The practical value of these applied problems formed the topicality and urgency of our investigation.

Finally, let us note that the qualitative theory of hypersingular integral equations and the numerical methods for solving them are developing and are of interest. For example, see sources [1, 3, 4, 5, 6, 13, 26, 27, 28, 29] and their references. Note that papers [3] and [5] are close to our study.

3 Function spaces, operators, and their properties

Let Π^I and Π^{II} be the two Hilbert spaces of polynomials with respect to the following inner products:

$$(u,v)_{\Pi^I}=\int_{-1}^{1} u(t)\,\overline{v(t)}\sqrt{1-t^2}dt+\int_{-1}^{1}\left(u(t)\sqrt{1-t^2}\right)'\overline{\left(v(t)\sqrt{1-t^2}\right)'}\sqrt{1-t^2}dt,$$

$$(u,v)_{\Pi^{II}}=\int_{-1}^{1} u(y)\,\overline{v(y)}\sqrt{1-y^2}dy. \tag{3.1}$$

Denote by T_n the first-kind Chebyshev polynomial of degree n; let the function U_{n-1} be the second-kind Chebyshev polynomial of degree $(n-1)$; let the set $\left\{t_{0j}^n\right\}_{j=1}^{n-1}$ be the set of roots of the polynomial U_{n-1} and $t_{0j}^n=\cos\frac{j}{n}\pi$ for $j=\overline{1,n-1}$.

The set $\left\{\sqrt{\frac{2}{\pi}}\frac{U_{m-1}(t)}{\sqrt{1+m^2}}\right\}_{m=1}^{n}$ forms an orthonormal system in the space Π^I and the set $\left\{\sqrt{\frac{2}{\pi}}U_{m-1}(y)\right\}_{m=1}^{n}$ is an orthonormal system in the space Π^{II}.

These sets are the bases of the spaces Π^I and Π^{II} respectively. Denote by $u_{n-2}^I(t)$ a polynomial of degree $(n-2)$ from the space Π^I. Then, this polynomial can be represented in the form $u_{n-2}^I(t) = \sum_{m=1}^{n-1} c_m^I \sqrt{\frac{2}{\pi}} \frac{U_{m-1}(t)}{\sqrt{1+m^2}}$, where $c_m^I = \left(u_{n-2}^I, \sqrt{\frac{2}{\pi}} \frac{U_{m-1}}{\sqrt{1+m^2}}\right)_{\Pi^I}$. Denote by $u_{n-2}^{II}(y)$ a polynomial of degree $(n-2)$ from the space Π^{II}. Then, this polynomial can be represented in the form $u_{n-2}^{II}(y) = \sum_{m=1}^{n-1} c_m^{II} \sqrt{\frac{2}{\pi}} U_{m-1}(y)$, where $c_m^{II} = \left(u_{n-2}^{II}, \sqrt{\frac{2}{\pi}} U_{m-1}\right)_{\Pi^{II}}$.

Now note that inner products (3.1) have an extra form of sum. For any pair of polynomials u_{n-2}^I and v_{n-2}^I of the space Π^I, we have that $\left(u_{n-2}^I, v_{n-2}^I\right)_{\Pi^I} = \sum_{m=1}^{n-1} c_m^I \overline{d_m^I}$, where the numbers c_m^I and d_m^I are the coefficients of the series for the functions respectively. In the same way, for any pair of polynomials of the space Π^{II}, we have $\left(u_{n-2}^{II}, v_{n-2}^{II}\right)_{\Pi^{II}} = \sum_{m=1}^{n-1} c_m^{II} \overline{d_m^{II}}$. Also, for the norms now we have similar extra forms. For any function u_{n-2}^I of the space Π^I, we have $\left\|u_{n-2}^I\right\|_{\Pi^I} = \sqrt{\sum_{m=1}^{n-1} |c_m^I|^2}$ and for any function u_{n-2}^{II} of the space Π^{II}, we have $\left\|u_{n-2}^{II}\right\|_{\Pi^{II}} = \sqrt{\sum_{m=1}^{n-1} |c_m^{II}|^2}$.

Let us introduce the following linear operators:

$$(Ru)(y) = u(y)\sqrt{1-y^2}, (Au)(y) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{(t-y)^2}\sqrt{1-t^2}dt,$$

$$\left(\Gamma^{-1}u\right)(y) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{y-t}\sqrt{1-t^2}dt, (Bu)(y) = \frac{1}{\pi}\int_{-1}^{1} \ln|t-y|\, u(t)\sqrt{1-t^2}dt,$$

$$(\mathbb{K}u)(y) = \frac{1}{\pi}\int_{-1}^{1} K(t,y)\, u(t)\sqrt{1-t^2}dt.$$

It is known that the operator A takes a second-kind Chebyshev polynomial to a similar polynomial and preserves its degree; the operator Γ^{-1} takes a polynomial to polynomial too and increasing its degree by one; the operator B takes a polynomial to a polynomial and increasing its degree by two. The operators A, Γ^{-1}, and B are defined on all of the space Π^I and acting to the space Π^{II}. So, for any natural number m we have the following rules:

$$A : U_{m-1}(t) \to -mU_{m-1}(y), \;\; \Gamma^{-1} : U_{m-1}(t) \to T_m(y),$$

$$B : U_0(t) \to \frac{1}{2}\left(\frac{T_2(y)}{2} - \ln 2\right) \text{ if } m = 1,$$

$$B : U_{m-1}(t) \to \frac{1}{2}\left(\frac{T_{m+1}(y)}{m+1} - \frac{T_{m-1}(y)}{m-1}\right) \text{ for } m \geq 2.$$

The operators R and $\mathbb{K}$ take the polynomials to the general-form functions. Let the function u_{n-2}^I belongs to the space Π^I as above. Then the operators A, Γ^{-1}, and B have the following form in the space Π^{II}:

$$\left(Au_{n-2}^I\right)(y) = \sum_{m=1}^{n-1} c_m^I \frac{-m}{\sqrt{1+m^2}} \sqrt{\frac{2}{\pi}} U_{m-1}(y),$$

$$\left(\Gamma^{-1}u^I_{n-2}\right)(y) = \sum_{m=1}^{n}\left(\Gamma^{-1}u^I_{n-1}, \sqrt{\frac{2}{\pi}}U_{m-1}\right)_{\Pi^{II}}\sqrt{\frac{2}{\pi}}U_{m-1}(y), \tag{3.2}$$

$$\left(Bu^I_{n-2}\right)(y) = \sum_{m=1}^{n+1}\left(Bu^I_{n-2}, \sqrt{\frac{2}{\pi}}U_{m-1}\right)_{\Pi^{II}}\sqrt{\frac{2}{\pi}}U_{m-1}(y). \tag{3.3}$$

These Π^{II}-representations of operators are important base points for our numerical method for solving hypersingular equation (2.1).

Further note that the operator A is invertible. If a function g belongs to the set $C^{0,\alpha}_{[-1,1]}$, then the equation $Au = g$ is solvable in a unique way and the unknown function $u(t) = -\frac{1}{\pi}\int_{-1}^{1}\frac{1}{y-t}\int_{-1}^{y} g(\tau)\,d\tau\frac{dy}{\sqrt{1-y^2}}$, where the outer integral is singular. The invert operator

$$\left(A^{-1}g\right)(t) = -\frac{1}{\pi}\int_{-1}^{1}\frac{1}{y-t}\int_{-1}^{y} g(\tau)\,d\tau\frac{dy}{\sqrt{1-y^2}}$$

takes the polynomial $U_{m-1}(y)$ to the polynomial $-\frac{1}{m}U_{m-1}(t)$, and acts from the space Π^{II} to the space Π^I. Since a function u^{II}_{n-2} belongs to the space Π^I, we have that the operator A^{-1} has the following form in the space Π^I:

$$\left(A^{-1}u^{II}_{n-2}\right)(t) = \sum_{m=1}^{n-1} c^{II}_m \frac{-\sqrt{1+m^2}}{m}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{1+m^2}}U_{m-1}(t).$$

Complete hypersingular integral equation of the second kind (2.1) in operator terms has the following form:

$$h(Ru)(y) - (Au)(y) + a\left(\Gamma^{-1}u\right)(y) + b(Bu)(y) + (Ku)(y) = f(y). \tag{3.4}$$

A solution of Equation (3.4) is called an exact solution and the equation is called the equation for an exact solution.

Denote by L^I and L^{II} the supplements of Hilbert spaces Π^I and Π^{II} with respect to the norms generated by corresponding inner products (3.1). The extensions of defined above linear operators to the introduced spaces L^I and L^{II} are denoted by the same symbols. The orthonormal systems $\left\{\sqrt{\frac{2}{\pi}}\frac{U_{m-1}(t)}{\sqrt{1+m^2}}\right\}_{m=1}^{\infty}$ and $\left\{\sqrt{\frac{2}{\pi}}U_{m-1}(y)\right\}_{m=1}^{\infty}$ are the bases of the spaces L^I and L^{II} respectively. Then, for any function $u^I(t)$ of the space L^I we have following representation: $u^I(t) = \sum_{m=1}^{\infty} c^I_m\sqrt{\frac{2}{\pi}}\frac{U_{m-1}(t)}{\sqrt{1+m^2}}$, where the coefficient $c^I_m = \left(u^I, \sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right)_{\Pi^I}$; for any function $u^{II}(y)$ of the space L^{II} we have the following representation: $u^{II}(y) = \sum_{m=1}^{\infty} c^{II}_m\sqrt{\frac{2}{\pi}}U_{m-1}(t)$, where the coefficient $c^{II}_m = \left(u^{II}, \sqrt{\frac{2}{\pi}}U_{m-1}\right)_{\Pi^{II}}$.

Now and as above we have the extra forms of sum for inner products in the spaces L^I and L^{II}. So, for any functions u^I and v^I of the space L^I, we have $\left(u^I, v^I\right)_{L^I} = \sum_{m=1}^{\infty} c^I_m\overline{d^I_m}$, where the numbers c^I_m and d^I_m are the coefficients of the series. By the way, for any pair of functions u^{II} and v^{II} of the space

L^{II}, we have $\left(u^{II},v^{II}\right)_{L^{II}}=\sum_{m=1}^{\infty}c_m^{II}\overline{d_m^{II}}$. The norms have the extra forms of the sum too. We have $\left\|u^I\right\|_{L^I}=\sqrt{\sum_{m=1}^{\infty}\left|c_m^I\right|^2}$ for any function u^I of the space L^I and $\left\|u^{II}\right\|_{L^{II}}=\sqrt{\sum_{m=1}^{\infty}\left|c_m^{II}\right|^2}$ for any function u^{II} of the space L^{II}.

As mentioned above the operators R and $\mathbb{K}$ take the polynomials of the space Π^I to the general-form functions of the space L^{II}. Let u_{n-1}^I be a polynomial from the space Π^I as above. Then the operators R and $\mathbb{K}$ have the following form in the space L^{II}:

$$\left(Ru_{n-1}^I\right)(y)=\sum_{m=1}^{\infty}\left(Ru_{n-1}^I,\sqrt{\frac{2}{\pi}}U_{m-1}\right)_{L^{II}}\sqrt{\frac{2}{\pi}}U_{m-1}(y), \tag{3.5}$$
$$\left(\mathbb{K}u_{n-1}^I\right)(y)=\sum_{m=1}^{\infty}\left(\mathbb{K}u_{n-1}^I,\sqrt{\frac{2}{\pi}}U_{m-1}\right)_{L^{II}}\sqrt{\frac{2}{\pi}}U_{m-1}(y).$$

The Π^{II}- and L^{II}-representations of operators are good optics for looking at their properties. Let us consider and collect them. Let us start with the operators A and A^{-1}. Since

$$\left\|Au_{n-1}^I\right\|_{\Pi^{II}}^2=\sum_{m=1}^{n}\tfrac{m^2}{1+m^2}\left|c_m^I\right|^2\leqslant\sum_{m=1}^{n}\left|c_m^I\right|^2=\left\|u_{n-1}^I\right\|_{\Pi^I}^2,$$

we have $\left\|Au_{n-1}^I\right\|_{\Pi^{II}}^2\leqslant 1$. We clearly have $\left\|A\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^{II}}^2=\frac{m^2}{1+m^2}$ and $\left\|\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^I}^2=1$. Then the ratio of numbers $\left\|A\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^{II}}^2$ and $\left\|\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^I}^2$ is equal to $\frac{m^2}{1+m^2}\xrightarrow[m\to\infty]{}1$. Thus, we obtain $\|A\|_{L^I\to L^{II}}=1$. By the way and for the operator A^{-1}, from the chain $\left\|A^{-1}u_{n-1}^{II}\right\|_{\Pi^I}^2=\sum_{m=1}^{n}\frac{1+m^2}{m^2}\left|c_m^{II}\right|^2\leqslant 2\left\|u_{n-1}^I\right\|_{\Pi^I}^2$, it follows that we have $\left\|A^{-1}u_{n-1}^{II}\right\|_{\Pi^I}^2\leqslant 2$. Finally, since $\left\|A^{-1}\sqrt{\frac{2}{\pi}}U_0\right\|_{L^I}^2=2$, we see that $\left\|A^{-1}\right\|_{L^{II}\to L^I}=\sqrt{2}$.

Let us move to the operators Γ^{-1} and B. It is easy to see that the operator Γ^{-1} is anti-self-adjoint and the operator B is self-adjoint. Using the properties of adjoint, Cauchy-Bunyakovsky inequality, sum form for norms, and equalities $\left\|\Gamma^{-1}\sqrt{\frac{2}{\pi}}U_0\right\|_{L^{II}}^2=\frac{1}{8}$, $\left\|\Gamma^{-1}\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^{II}}^2=\frac{1}{2}\frac{1}{1+m^2}$ for $m\geqslant 2$, $\left\|B\sqrt{\frac{2}{\pi}}U_0\right\|_{L^{II}}^2=\frac{1}{64}\left(8\ln^2 2+4\ln 2+1\right)$, $\left\|B\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^{II}}^2=\frac{17}{720}$ if $m=2$, and $\left\|B\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^{II}}^2=\frac{1}{8}\frac{1}{1+m^2}\left(\frac{1}{(m-1)^2}+\frac{1}{m^2-1}+\frac{1}{(m+1)^2}\right)$ for $m\geqslant 3$, we get that the operators Γ^{-1} and B are bounded too. We will bit touch on this point in Section 4.

It is obvious that the operator R is bounded. So, we have $\left\|Ru^I\right\|_{L^{II}}^2\leqslant\int_{-1}^{1}\left|\sqrt{1-y^2}u^I(y)\right|^2\sqrt{1-y^2}dy\leqslant\max_{-1<y<1}\left(1-y^2\right)\int_{-1}^{1}\left|u^I(y)\right|^2\sqrt{1-y^2}dy\leqslant\left\|u^I\right\|_{L^I}^2$. It now follows that $\left\|R\sqrt{\frac{2}{\pi}}U_0\right\|_{L^{II}}^2=\frac{3}{8}$ and $\left\|R\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^{II}}^2=$

$\frac{1}{2}\frac{1}{1+m^2}$ for $m \geqslant 2$. The operator $\mathbb{K}$ is bounded too; we have the following estimations: $\left\|\mathbb{K}u^I\right\|^2_{L^{II}} \leqslant \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\left|K\left(t,y\right)\right|^2\sqrt{1-t^2}dt\sqrt{1-y^2}dy\left\|u^I\right\|^2_{L^I}$ and $\left\|\mathbb{K}\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|^2_{L^{II}} \leqslant \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\left|K\left(t,y\right)\right|^2\sqrt{1-t^2}dt\sqrt{1-y^2}dy\frac{1}{1+m^2}$ for $m \geqslant 1$.

The main idea of the developed numerical method for integral equation (3.4) is the assumption that a solution has a polynomial form. The main task of the development is the left-hand side of Equation (3.4) takes a polynomial to a general-form function. Therefore, the main task is, as a possible minimum in the L^{II}-norm sense, to change Equation (3.4) that the left-hand side takes a polynomial to polynomial and preserves its degree. We shall say that the changed integral equation is called the equation for an approximate solution, and the change procedure is called regularization.

4 Interpolation polynomial and regularization of operators

As mentioned above the operators R, A, Γ^{-1}, B, and $\mathbb{K}$ take the elements of the space Π^I to the different functions. This is an obstacle to the polynomial discretization of the complete hypersingular integral equation of the second kind. For eliminating this obstacle, the operators R, Γ^{-1}, B, and $\mathbb{K}$ were regularized. We say that the regularization of an operator is called an operator that preserves the degree of a polynomial and is close to the regularized operator in the norm. As we know, this definition was first given to the regularization of the operator by Prof. Ivan Lifanov in [21]. This definition has become widespread. It was used in key works [8,12,14,20]. Now the term regularization of operator continues to be used in this sense.

Let the function u_{n-2} be a polynomial of degree $(n-2)$. Then, the interpolation polynomial of the function u_{n-2} has following form: $u_{n-2}\left(t\right) = \sum_{j=1}^{n-1} u_{n-2}\left(t^n_{0j}\right) l_{n-2,j}\left(t\right)$, where the function $l_{n-2,j}\left(t\right) = \frac{U_{n-1}(t)}{U'_{n-1}\left(t^n_{0j}\right)\left(t-t^n_{0j}\right)}$ for $j=\overline{1,n-1}$ and are the barycentric form of the Lagrange basis polynomial.

For now, let u be a general-form function and u_{n-2} be the interpolation polynomial of the function u. Further, we shall denote the interpolation polynomial of a function by an intuitive subscript corresponding to the polynomial degree. As this is shown in the previous sentence. Let us estimate in the space L^{II} the deviation of the interpolation polynomial from a function u that belongs to the set $C^{r,\alpha}_{[-1,1]}$. Denote by E_{n-2} the least deviation of $(n-2)$-degree polynomials of the space Π^{II} from the function u. So, we have $E_{n-2} = \inf_{u_{n-2}\in\Pi^I}\max_{t\in[-1,1]}\left|u_{n-2}\left(t\right)-u\left(t\right)\right|$. As shown in [23], we have Jackson's inequality for the value E_{n-2} and $E_{n-2} \leqslant \frac{c_u}{(n-2)^{r+\alpha}}$, where the number n is at least $r+3$ and the constant c_u is n-independent. From [23], we have that there exists a polynomial $\widetilde{u}_{n-2}\left(t\right)$ such that $\max_{t\in[-1,1]}\left|\widetilde{u}_{n-2}\left(t\right)-u\left(t\right)\right| = E_{n-2}$. Using the parallelogram inequality, we have $\left\|u_{n-2}-u\right\|_{L^{II}} \leqslant \left\|\widetilde{u}_{n-2}-u\right\|_{L^{II}} + \left\|u_{n-2}-\widetilde{u}_{n-2}\right\|_{L^{II}}$. For the first term of the inequality right-hand side, we clearly have $\left\|\widetilde{u}_{n-2}-u\right\|^2_{L^{II}} = \int_{-1}^{1}\left|\widetilde{u}_{n-2}\left(y\right)-u\left(y\right)\right|^2\sqrt{1-y^2}dy \leqslant \frac{\pi}{2}E^2_{n-2}$. For the second term, using the function $u_{n-2}-\widetilde{u}_{n-2}$ is the $(n-2)$-degree polynomial and is equal to $\sum_{j=1}^{n-1}\left(u\left(t_{0j}^{n}\right)-\widetilde{u}_{n-2}\left(t_{0j}^{n}\right)\right)l_{n-2,j}\left(t\right)$, and the rule for the Lagrange basis polynomials $\left(l_{n-2,j_1},l_{n-2,j_2}\right)_{L^{II}}=\delta_{j_1j_2}\left(1-\left(t_{0j_1}^{n}\right)^2\right)$, we have the chain $\left\|u_{n-2}-\widetilde{u}_{n-2}\right\|_{L^{II}}^{2}=\frac{\pi}{n}\sum_{j=1}^{n-1}\left|u\left(t_{0j}^{n}\right)-\widetilde{u}_{n-2}\left(t_{0j}^{n}\right)\right|^{2}\left(1-\left(t_{0j}^{n}\right)^2\right)\leqslant\pi E_{n-2}^{2}$. Thus, we obtain the following inequality:

$$\left\|u_{n-2}-u\right\|_{L^{II}}\leqslant 2\sqrt{\pi}c_u/\left(n-2\right)^{r+\alpha}, \tag{4.1}$$

where the number n is at least $r+3$ and the constant c_u is n-independent.

Let us shift to the regularization of operators. Denote by R_{n-2} the regularization of operator R. Since the operator R has L^{II}-form (3.5), the operator R_{n-2} has the following form:

$$\left(R_{n-2}u_{n-2}\right)\left(y\right)=\left(Ru_{n-2}\right)\left(y\right)-\sum_{k=n}^{\infty}\left(Ru_{n-2},\sqrt{\frac{2}{\pi}}U_{k-1}\right)_{L^{II}}\sqrt{\frac{2}{\pi}}U_{k-1}\left(y\right)$$

and $\left(R_{n-2}u_{n-2}\right)\left(y\right)=\sum_{k=1}^{n-1}\left(Ru_{n-2},\sqrt{\frac{2}{\pi}}U_{k-1}\right)_{L^{II}}\sqrt{\frac{2}{\pi}}U_{k-1}\left(y\right)$. By construction, the operator R_{n-2} takes a polynomial of the space $\varPi^{I}$ to a polynomial of the space $\varPi^{II}$ and preserves its degree. Since the function $R_{n-2}u_{n-2}$ is a polynomial and the interpolation polynomial is unique, it follows that $\left(R_{n-2}u_{n-2}\right)\left(y\right)=\sum_{k=1}^{n-1}\sqrt{1-\left(t_{0k}^{n}\right)^2}u_{n-2}\left(t_{0k}^{n}\right)l_{n-2,k}\left(y\right)$. Note that we will use this interpolation form of the operator R_{n-2} for our numerical method. Now, let us estimate the L^{II}-space norm of the difference between operators R_{n-2} and R. We have $\left\|R_{n-2}u_{n-2}-Ru_{n-2}\right\|_{L^{II}}^{2}=\sum_{k=n}^{\infty}\left|\left(Ru_{n-2},\sqrt{\frac{2}{\pi}}U_{k-1}\right)_{L^{II}}\right|^{2}$. We need to estimate the value $\left|\left(Ru_{n-2},\sqrt{\frac{2}{\pi}}U_{k-1}\right)_{L^{II}}\right|$ at the top. By definition, $\left(Ru_{n-2},\sqrt{\frac{2}{\pi}}U_{k-1}\right)_{L^{II}}=\int_{-1}^{1}u_{n-2}\left(y\right)\sqrt{1-y^2}\sqrt{\frac{2}{\pi}}U_{k-1}\left(y\right)\sqrt{1-y^2}dy$. Using the way of integration by parts, we get that $\left(Ru_{n-2},\sqrt{\frac{2}{\pi}}U_{k-1}\right)_{L^{II}}=-\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\left(u_{n-2}\left(y\right)\sqrt{1-y^2}\right)'\left(\frac{U_k(y)}{k+1}-\frac{U_{k-2}(y)}{k-1}\right)\sqrt{1-y^2}dy$. By the Cauchy-Bunyakovsky inequality, we obtain the estimation $\left|\left(Ru_{n-2},\sqrt{\frac{2}{\pi}}U_{k-1}\right)_{L^{II}}\right|^{2}\leqslant\frac{1}{2}\left\|u_{n-2}\right\|_{L^{I}}^{2}\frac{1}{(k-2)^2}$. Thus, we clearly have the chain $\left\|R_{n-2}u_{n-2}-Ru_{n-2}\right\|_{L^{II}}^{2}\leqslant\frac{1}{2}\left\|u_{n-2}\right\|_{\varPi^{I}}^{2}\sum_{k=n}^{\infty}\frac{1}{(k-2)^2}=\frac{1}{2}\left\|u_{n-2}\right\|_{\varPi^{I}}^{2}\psi^{(1)}\left(n-2\right)\leqslant\frac{1}{n-2}\left\|u_{n-2}\right\|_{\varPi^{I}}^{2}$, whereby $\psi^{(1)}$ denoted the polygamma function of order 1 and we used the wiki-known two-sided estimation $\frac{1}{n-2}+\frac{1}{2(n-2)^2}\leqslant\psi^{(1)}\left(n-2\right)\leqslant\frac{1}{n-2}+\frac{1}{(n-2)^2}$ for all natural numbers n starting from three. Finally, we obtain the operator R_{n-2} is close to the operator R with respect to the L^{II}-space norm. There is the following inequality:

$$\left\|R_{n-2}-R\right\|_{\varPi^{I}\to L^{II}}\leqslant 1/\sqrt{n-2}, \tag{4.2}$$

where the number n is at least three.

Let Γ^{-1}_{n-2} be the regularization of operator Γ^{-1}. Since the operator Γ^{-1} has Π^{II}-form (3.2), the operator Γ^{-1}_{n-2} has the form

$$\left(\Gamma^{-1}_{n-2}u_{n-2}\right)(y) = \left(\Gamma^{-1}u_{n-2}\right)(y) - \left(\Gamma^{-1}u_{n-2}, \sqrt{\frac{2}{\pi}}U_{n-1}\right)_{L^{II}}\sqrt{\frac{2}{\pi}}U_{n-1}(y)$$

and acting from the space Π^{I} to the space Π^{II} preserving the degree of the polynomial. Since the operator Γ^{-1} is anti-self-adjoint and takes the polynomial U_{n-1} to T_n, it follows that $\left(\Gamma^{-1}u_{n-2}, \sqrt{\frac{2}{\pi}}U_{n-1}\right)_{L^{II}}\sqrt{\frac{2}{\pi}}U_{n-1}(y) = -\frac{2}{\pi}\int_{-1}^{1}u_{n-2}(t)\,T_n(t)\sqrt{1-t^2}dtU_{n-1}(y)$. By the way, for the operator Γ^{-1}_{n-2}, we have $\left(\Gamma^{-1}_{n-2}u_{n-2}\right)(y) = \frac{1}{\pi}\int_{-1}^{1}\left(\frac{1}{y-t}+2T_n(t)\,U_{n-1}(y)\right)u(t)\sqrt{1-t^2}dt$. We will use this integral form of the operator Γ^{-1}_{n-2} for the numerical method. Let us estimate the closeness of operators Γ^{-1}_{n-2} and Γ^{-1} with respect to the L^{II}-space norm. We have $\left\|\Gamma^{-1}_{n-2}u_{n-2} - \Gamma^{-1}u_{n-2}\right\|^2_{L^{II}} = \left|\left(\Gamma^{-1}u_{n-2}, \sqrt{\frac{2}{\pi}}U_{n-1}\right)_{L^{II}}\right|^2$. Let us estimate the value $\left|\left(\Gamma^{-1}u_{n-2}, \sqrt{\frac{2}{\pi}}U_{n-1}\right)_{L^{II}}\right|$ at the top. Following the way of the integration by parts and Cauchy-Bunyakovsky inequality, which we passed with the operator R_{n-2}, we obtain $\left|\left(\Gamma^{-1}u_{n-2}, \sqrt{\frac{2}{\pi}}U_{n-1}\right)_{L^{II}}\right|^2 \leqslant \frac{1}{2}\left\|u_{n-2}\right\|^2_{\Pi^{I}}\frac{1}{(n-2)^2}$. Thus, we have obtained the operator Γ^{-1}_{n-2} is close to the operator Γ^{-1} with inequality

$$\left\|\Gamma^{-1}_{n-2} - \Gamma^{-1}\right\|_{\Pi^{I}\to L^{II}} \leqslant \frac{\sqrt{2}}{2}\frac{1}{n-2}, \tag{4.3}$$

where the number n is at least three.

By B_{n-2} denote the regularization of operator B. Since the operator B has Π^{II}-form (3.3), the operator B_{n-2} has the following form:

$$\left(B_{n-2}u_{n-2}\right)(y) = \left(Bu_{n-2}\right)(y) - \left(Bu_{n-2}, \sqrt{\frac{2}{\pi}}U_{n-1}\right)_{L^{II}}\sqrt{\frac{2}{\pi}}U_{n-1}(y)$$
$$- \left(Bu_{n-2}, \sqrt{\frac{2}{\pi}}U_{n}\right)_{L^{II}}\sqrt{\frac{2}{\pi}}U_{n}(y).$$

The operator B_{n-2} is acting from the space Π^{I} to the space Π^{II} and takes the $(n-2)$-degree polynomial to the polynomial of the same degree. Since the operator B_{n-2} is self-adjoint and takes the polynomial U_{n-1} to the polynomial $\frac{1}{2}\left(\frac{T_{m+1}(y)}{m+1} - \frac{T_{m-1}(y)}{m-1}\right)$ for all natural numbers m without 1, it follows that the second term from the right-hand side of the representation of the operator B_{n-2} is equal to $\frac{1}{\pi}\int_{-1}^{1}u_{n-2}(t)\left(\frac{T_{n+1}(t)}{n+1} - \frac{T_{n-1}(t)}{n-1}\right)\sqrt{1-t^2}dtU_{n-1}(y)$ and the third term is $\frac{1}{\pi}\int_{-1}^{1}u_{n-2}(t)\left(\frac{T_{n+2}(t)}{n+2} - \frac{T_{n}(t)}{n}\right)\sqrt{1-t^2}dtU_{n}(y)$. So, by the way, for the operator B_{n-2}, we get the following integral form: $\left(B_{n-2}u_{n-2}\right)(y) = \frac{1}{\pi}\int_{-1}^{1}\Big(\ln|t-y| - \left(\frac{T_{n+1}(t)}{n+1} - \frac{T_{n-1}(t)}{n-1}\right)U_{n-1}(y) - \left(\frac{T_{n+2}(t)}{n+2} - \frac{T_{n}(t)}{n}\right)U_{n}(y)\Big)\cdot$

$u_{n-2}(t)\sqrt{1-t^2}dt$. We will use this form for the numerical method. Let us estimate the difference between operators B_{n-2} and B with respect to the L^{II}-space norm. We have $\|B_{n-2}u_{n-2}-Bu_{n-2}\|^2_{L^{II}}=\left|\left(Bu_{n-2},\sqrt{\frac{2}{\pi}}U_{n-1}\right)_{L^{II}}\right|^2+\left|\left(Bu_{n-2},\sqrt{\frac{2}{\pi}}U_{n}\right)_{L^{II}}\right|^2$. We need to estimate at the top the values of two terms from the right-hand side. Using the integration by parts and Cauchy-Bunyakovsky inequality, we get two inequalities $\left|\left(Bu_{n-2},\sqrt{\frac{2}{\pi}}U_{n-1}\right)_{L^{II}}\right|^2\leqslant\frac{3}{8}\|u_{n-2}\|^2_{\Pi^I}\frac{1}{(n-2)^4}$ and $\left|\left(Bu_{n-2},\sqrt{\frac{2}{\pi}}U_{n}\right)_{L^{II}}\right|^2\leqslant\frac{3}{8}\|u_{n-2}\|^2_{\Pi^I}\frac{1}{(n-2)^4}$. Finally, we have obtained that the operator B_{n-2} is close to the operator B and the following inequality is true:

$$\|B_{n-2}-B\|_{\Pi^I\to L^{II}}\leqslant\frac{\sqrt{3}}{2}\frac{1}{(n-2)^2}, \tag{4.4}$$

where the number n is at least three.

Let the function K_{n-2} be the $(n-2)$-degree interpolation polynomial of the two-variables function K with the nod set $\left\{t^n_{0j}\right\}_{j=1}^{n-1}$ and with respect to each variable. Then, we have equalities $K_{n-2}\left(t^n_{0j},t^n_{0k}\right)=K\left(t^n_{0j},t^n_{0k}\right)$ for $j=\overline{1,n-1}$ and $k=\overline{1,n-1}$. By $K_{n-2,1}$ denote the $(n-2)$-degree interpolation polynomial for the first variable of the function K with the nod set $\left\{t^n_{0j}\right\}_{j=1}^{n-1}$ and by $K_{n-2,1}$ denote the $(n-2)$-degree interpolation polynomial for the second variable. Denote by $\mathbb{K}_{n-2}$ the regularization of operator $\mathbb{K}$ and for $(n-2)$-degree polynomial from the space Π^I, we have

$$\left(\mathbb{K}_{n-2}u_{n-2}\right)(y)=\frac{1}{\pi}\int_{-1}^{1}K_{n-2}(t,y)\,u_{n-2}(t)\sqrt{1-t^2}dt.$$

The operator $\mathbb{K}_{n-2}$ takes a polynomial to a polynomial and preserves its degree. Let us show that the operator $\mathbb{K}_{n-2}$ is close to operator $\mathbb{K}$ with respect to the L^{II}-space norm. Since the function K belongs to the set $C^{1,\alpha}_{[-1,1]}$ in each variable uniformly with respect to another one, we have two following inequalities by estimation (4.1):

$$\left(\int_{-1}^{1}\left|K_{n-2,1}(t,y)-K(t,y)\right|^2\sqrt{1-t^2}dt\right)^{1/2}\leqslant\frac{2\sqrt{\pi}c_K}{(n-2)^{1+\alpha}}, \tag{4.5}$$
$$\left(\int_{-1}^{1}\left|K_{n-2,2}(t,y)-K(t,y)\right|^2\sqrt{1-y^2}dy\right)^{1/2}\leqslant\frac{2\sqrt{\pi}c_K}{(n-2)^{1+\alpha}},$$

where the number n is at least three, the constant c_K is independent of n, and the constant α is the Hölder exponent and greater than zero. Using the Cauchy-Bunyakovsky inequality, we get the estimation $\|\mathbb{K}_{n-2}u_{n-2}-\mathbb{K}u_{n-2}\|_{L^{II}}\leqslant\frac{1}{\pi}\sqrt{\int_{-1}^{1}\int_{-1}^{1}\left|K_{n-2}(t,y)-K(t,y)\right|^2\sqrt{1-t^2}dt\sqrt{1-y^2}dy}\|u_{n-2}\|_{L^I}$. Let us add and subtract the interpolation polynomial $K_{n-2,2}$ from the function $K_{n-2}-K$. By the triangle inequality, we get that the cumbersome second factor of the estimation is less than or equal to the sum of two following no less cumbersome terms: $\sqrt{\int_{-1}^{1}\left(\int_{-1}^{1}\left|K_{n-2,1}\left(t,y\right)-K\left(t,y\right)\right|^{2}\sqrt{1-t^{2}}dt\right)_{n-2,2}\sqrt{1-y^{2}}dy}$ and $\sqrt{\int_{-1}^{1}\int_{-1}^{1}\left|K_{n-2,2}\left(t,y\right)-K\left(t,y\right)\right|^{2}\sqrt{1-t^{2}}dt\sqrt{1-y^{2}}dy}$. Remind that the subscript is related to interpolation and denotes the degree of the polynomial and the number of variable. Also, the first term is based on the rule $\int_{-1}^{1}\left|u_{n-2}\left(t\right)\right|^{2}\sqrt{1-t^{2}}dt = \int_{-1}^{1}\left(\left|u\left(t\right)\right|^{2}\right)_{n-2}\sqrt{1-t^{2}}dt$. Using the quadrature formula $\int_{-1}^{1}u_{n-2}\left(t\right)\sqrt{1-t^{2}}dt = \frac{\pi}{n}\sum_{j=1}^{n-1}u_{n-2}\left(t_{0j}^{n}\right)\left(1-\left(t_{0j}^{n}\right)^{2}\right)$ and inequality (4.5), we obtain that the first term is equal to the cumbersome expression $\sqrt{\frac{\pi}{n}\sum_{j=1}^{n-1}\left(\int_{-1}^{1}\left|K_{n-2,1}\left(t,t_{0j}^{n}\right)-K\left(t,t_{0j}^{n}\right)\right|^{2}\sqrt{1-t^{2}}dt\right)_{n-2,2}\left(1-\left(t_{0j}^{n}\right)^{2}\right)}$ and is not greater than the miniature fraction $\frac{2\pi c_K}{(n-2)^{1+\alpha}}$. To estimate the second term, we use inequality (4.5). So, the second term is bounded by the fraction $\frac{\pi c_K}{(n-2)^{1+\alpha}}$. Thus, we obtain that the operator $\mathbb{K}_{n-2}$ is close to the operator $\mathbb{K}$ with respect to the L^{II}-space norm. There is the following inequality:

$$\left\|\mathbb{K}_{n-2}-\mathbb{K}\right\|_{\Pi^{I}\to L^{II}} \leqslant 3c_K/(n-2)^{1+\alpha}, \tag{4.6}$$

where the number n is at least three, the constant c_K is independent of n, and the constant α is the Hölder exponent and greater than zero.

Thus, the operators R_{n-2}, Γ_{n-2}^{-1}, B_{n-2}, and $\mathbb{K}_{n-2}$ are defined as the result of the regularization procedure. All of them are acting from the space Π^{I} to the space Π^{II}, preserve a polynomial degree, and are closed in norm to the operators R, Γ^{-1}, B, and $\mathbb{K}$ respectively. We have that the left-hand side of Equation (3.4) is boundedly changed with estimates (4.2)–(4.6). It takes a polynomial to a polynomial and preserves its degree.

Note that there are other regularization methods. For example, in the book [24], an interpolation polynomial was used to regularize the operator.

Denote by $f_{n-2}\left(y\right)$ the $(n-2)$-degree interpolation polynomial of the function $f\left(y\right)$ with nod set $\left\{t_{0j}^{n}\right\}_{j=1}^{n-1}$. Then, we have equalities $f_{n-2}\left(t_{0j}^{n}\right) = f\left(t_{0j}^{n}\right)$ for $j=\overline{1,n-1}$. Since estimation (4.1), the following inequality is true:

$$\left\|f_{n-2}-f\right\|_{L^{II}} \leqslant \frac{2\sqrt{\pi}c_f}{(n-2)^{\alpha}}, \tag{4.7}$$

where the number n is at least three, the constant c_f is independent of n, and the constant α is the Hölder exponent and greater than zero.

Finally, the equation for an approximate solution is defined as the result of the regularization procedure. It is the regularized complete hypersingular integral equation of the second kind and it has the following operator form:

$$\begin{aligned}&h\left(R_{n-2}u_{n-2}\right)(y)-\left(Au_{n-2}\right)(y)\\&+a\left(\Gamma_{n-2}^{-1}u_{n-2}\right)(y)+b\left(B_{n-2}u_{n-2}\right)(y)\\&+\left(\mathbb{K}_{n-2}u_{n-2}\right)(y)=f_{n-2}\left(y\right).\end{aligned} \tag{4.8}$$

5 Discretization: quadrature formulae and system of linear algebraic equations

This section is about the discretization of Equation (4.8) for an approximate solution. Using quadrature formulae of interpolation type by the Nyström way, we transform it to a system of linear algebraic equations. Suppose that a solution of Equation (4.8) has the form of $(n-2)$-degree polynomial. Then, the left-hand and right-hand sides of Equation (4.8) are polynomials of degree $(n-2)$. The coincidence of these polynomials at $(n-1)$ different points is necessary and sufficient for the identity of the left-hand and right-hand sides of Equation (4.8). Assume that y is equal to t_{0j}^n for $j=\overline{1,n-1}$. Then, Equation (4.8) is transformed to $(n-1)$ equalities

$$\begin{aligned} & h\left(R_{n-2}u_{n-2}\right)\left(t_{0j}^n\right)-\left(Au_{n-2}\right)\left(t_{0j}^n\right)+a\left(\mathit{\Gamma}_{n-2}^{-1}u_{n-2}\right)\left(t_{0j}^n\right) \\ & +b\left(B_{n-2}u_{n-2}\right)\left(t_{0j}^n\right)+\left(\mathbb{K}_{n-2}u_{n-2}\right)\left(t_{0j}^n\right)=f_{n-2}\left(t_{0j}^n\right), \text{ for } j=\overline{1,n-1}. \end{aligned} \tag{5.1}$$

Now let us reduce system (5.1) to the system of linear algebraic equations. We use the quadrature formulae of interpolation type [9, 12]. These formulae are representations of the operators R_{n-2}, A, $\mathit{\Gamma}_{n-2}^{-1}$, B_{n-2}, and $\mathbb{K}_{n-2}$ in the polynomial space $\mathit{\Pi}^{II}$. For the operator R_{n-2}, we have the following formula:

$$\left(R_{n-2}u_{n-2}\right)\left(t_{0j}^n\right)=\sum_{k=1}^{n-1}a_{jk}^{(1)}u_{n-2}\left(t_{0k}^n\right), \text{ for } j=\overline{1,n-1}, \tag{5.2}$$

where the coefficient $a_{jk}^{(1)}=0$ for $k\neq j$ and $a_{jk}^{(1)}=\sqrt{1-\left(t_{0k}^n\right)^2}$ for $k=j$, and $k=\overline{1,n-1}$. For the operator A we have

$$\left(Au_{n-2}\right)\left(t_{0j}^n\right)=\sum_{k=1}^{n-1}a_{jk}^{(2)}u_{n-2}\left(t_{0k}^n\right), \text{ for } j=\overline{1,n-1},$$

where the coefficient $a_{jk}^{(2)}=\frac{1}{n}\left(1-\left(t_{0k}^n\right)^2\right)\frac{(-1)^{k+j+1}+1}{\left(t_{0j}^n-t_{0k}^n\right)^2}$ for $k\neq j$ and $a_{jk}^{(2)}=-\frac{n}{2}$ for $k=j$, and $k=\overline{1,n-1}$. For the operator $\mathit{\Gamma}_{n-2}^{-1}$ we have

$$\left(\mathit{\Gamma}_{n-2}^{-1}u_{n-2}\right)\left(t_{0j}^n\right)=\sum_{k=1}^{n-1}a_{jk}^{(3)}u_{n-2}\left(t_{0k}^n\right), \text{ for } j=\overline{1,n-1},$$

where the coefficient $a_{jk}^{(3)}=\frac{1}{n}\left(1-\left(t_{0k}^n\right)^2\right)\frac{(-1)^{j+k+1}+1}{t_{0j}^n-t_{0k}^n}$ for $k\neq j$ and $a_{jk}^{(3)}=0$ for $k=j$, and $k=\overline{1,n-1}$. The quadrature formula is different too. Let us show the way of this quadrature formula of interpolation type. The derivation of other formulae goes similarly.

So, we have $u_{n-2}(t)=\sum_{j=1}^{n-1}u_{n-2}\left(t_{0j}^n\right)l_{n-2,j}(t)$ and $\left(\mathit{\Gamma}_{n-2}^{-1}u_{n-2}\right)(y)=\sum_{j=1}^{n-1}u_{n-2}\left(t_{0j}^n\right)\left(\mathit{\Gamma}_{n-2}^{-1}l_{n-2,j}\right)(y)$. Further, it is easy to see that $\left(\mathit{\Gamma}_{n-2}^{-1}l_{n-2,j}\right)(y)=\frac{1}{n}(-1)^j\left(1-\left(t_{0j}^n\right)^2\right)\frac{T_n(y)-T_n\left(t_{0j}^n\right)}{y-t_{0j}^n}+\frac{2}{n}(-1)^j\left(1-\left(t_{0j}^n\right)^2\right)U_{n-1}(y)$. Using

L'Hospital's rule in a neighborhood of the point t_{0j}^n, we get $\left(\Gamma_{n-2}^{-1} l_{n-2,j}\right)\left(t_{0j}^n\right) = 0$ for $j=\overline{1,n-1}$. If $y=t_{0k}^n$ and $k \neq j$, then

$$\left(\Gamma_{n-2}^{-1} l_{n-2,j}\right)\left(t_{0k}^n\right) = \tfrac{1}{n}\left(1-\left(t_{0k}^n\right)^2\right) \tfrac{(-1)^{j+k+1}+1}{t_{0j}^n - t_{0k}^n}.$$

For the operator B_{n-2} we have

$$\left(B_{n-2}u_{n-2}\right)\left(t_{0j}^n\right) = \sum_{k=1}^{n-1} a_{jk}^{(4)} u_{n-2}\left(t_{0k}^n\right), \text{ for } j=\overline{1,n-1},$$

where the coefficient $a_{jk}^{(4)} = -\frac{1}{n}\left(1-\left(t_{0k}^n\right)^2\right)\left(\ln 2 + 2\sum_{M=1}^{n-1} \frac{T_M\left(t_{0k}^n\right)T_M\left(t_{0j}^n\right)}{M} + \frac{2(-1)^{j+k}}{n} - \frac{(-1)^j T_{n+2}\left(t_{0k}^n\right)}{n+2}\right)$ for $k=\overline{1,n-1}$. For the operator $\mathbb{K}_{n-2}$ we have

$$\left(\mathbb{K}_{n-2}u_{n-2}\right)\left(t_{0j}^n\right) = \sum_{k=1}^{n-1} a_{jk}^{(5)} u_{n-2}\left(t_{0k}^n\right), \text{ for } j=\overline{1,n-1}, \tag{5.3}$$

where the coefficient $a_{jk}^{(5)} = \frac{1}{n}\left(1-\left(t_{0k}^n\right)^2\right) K_{n-2}\left(t_{0j}^n, t_{0k}^n\right)$ for $k=\overline{1,n-1}$.

After transformation by quadrature formulae (5.2)–(5.3), the left-hand side of system (5.1) will take the form of a linear combination of the values of the unknown function at the interpolation nodes $u_{n-2}\left(t_{0j}^n\right)$, $j=\overline{1,n-1}$. Thus, we obtain the system of linear algebraic equations for the vector $\left(u_{n-2}\left(t_{0j}^n\right)\right)_{j=1}^{n-1}$ and that has form

$$\sum_{k=1}^{n-1} a_{jk} u_{n-2}\left(t_{0k}^n\right) = f_{n-2}\left(t_{0j}^n\right), \text{ for } j=\overline{1,n-1}, \tag{5.4}$$

where the coefficients $a_{jk} = h a_{jk}^{(1)} - a_{jk}^{(2)} + a a_{jk}^{(3)} + b a_{jk}^{(4)} + a_{jk}^{(5)}$.

Note that the terms $a_{jk}^{(2)}$ for $k=j$ have the greatest value in the coefficients a_{jk}, $j=\overline{1,n-1}$, $k=\overline{1,n-1}$. The term $-\frac{n}{2}$ is included in each diagonal element of the matrix of system of equations (5.4). All other terms that form the coefficients of the matrix are bounded or decrease as the ratio $\frac{1}{n}$ with increasing the number n. Therefore, the diagonal elements exceed the other elements of the matrix rows respectively. This is a useful point and highlights the presented numerical method.

After system of linear algebraic equations (5.4) is solved and the components of the vector $\left(u_{n-2}\left(t_{0j}^n\right)\right)_{j=1}^{n-1}$ are obtained, an approximate solution of Equation (2.1) is constructed as follows: $u_{n-2}(t) = \sum_{j=1}^{n-1} u_{n-2}\left(t_{0j}^n\right) l_{n-2,j}(t)$.

Finally, we have the following summary theorem.

Theorem 1. *The interpolation polynomial constructed by the solution of system of linear algebraic equations* (5.4) *is an approximate solution of complete hypersingular integral equation of the second kind* (2.1).

6 Theorem on existence and uniqueness of a solution of the complete second-kind hypersingular integral equation

In the section, we shall prove an existence and uniqueness criterion for a solution of second-kind complete hypersingular integral Equation (2.1). To prove it, we need the following theorems.

Theorem 2. *The operator A acting from the space L^I to the space L^{II} is bounded with the equality $\|A\|_{L^I \to L^{II}} = 1$ and boundedly invertible, i.e., there exists the operator A^{-1} acting from the space L^{II} to the space L^I with the equality $\left\|A^{-1}\right\|_{L^{II} \to L^I} = \sqrt{2}$.*

Indeed, this theorem has already been proved in Section 3 and sums up the obtained results.

Further, we need equipment to characterize the compactness of operators.

Theorem 3. *Let the set $\{e_m\}_{m=1}^{\infty}$ be an orthonormal basis of a Hilbert space X, let Y be a Banach space, let Q be a bounded linear operator, and the series $\sum_{m=1}^{\infty} \|Qe_m\|_Y^2$ is convergent; then the operator Q is compact.*

Proof. The idea is to approximate the operator Q by a sequence of compact operators. Let us denote the sequence of operators by Q_n for all natural numbers n. We will construct the sequence of operators acting from the Hilbert space X to the Banach space Y based on the operator Q. Let us start by defining the operator Q_n on the basis $\{e_m\}_{m=1}^{\infty}$ of the space X. Let $Q_n e_m = Qe_m$ for $m \leqslant n$ and $Q_n e_m = 0$ for $m > n$. Now let us extend the operator Q_n to all elements of the space X. Denote by x any element of the space X. Then we have $x = \sum_{m=1}^{\infty} (x, e_m)_X e_m$ and $Q_n x = \sum_{m=1}^{n} (x, e_m)_X Q_n e_m$. So, we have constructed the sequence Q_n. All operators in the sequence are finite-dimensional. Hence all of them are compact.

It is clear, that the operator Q has the form $Qx = \sum_{m=1}^{\infty} (x, e_m)_X Qe_m$.

Let us estimate the value $\|Qx - Q_n x\|_Y$. Using Hölder's inequality for the series, we get the chain

$$\|Qx - Q_n x\|_Y = \Big\| \sum_{m=n+1}^{\infty} (x, e_m)_X Qe_m \Big\|_Y \leqslant \sum_{m=n+1}^{\infty} |(x, e_m)_X| \, \|Qe_m\|_Y$$
$$\leqslant \Big(\sum_{m=n+1}^{\infty} |(x, e_m)_X| \Big)^{1/2} \Big(\sum_{m=n+1}^{\infty} \|Qe_m\|_Y \Big)^{1/2}.$$

By Bessel's inequality, we have $\sum_{m=n+1}^{\infty} |(x, e_m)_X| \leqslant \sum_{m=1}^{\infty} |(x, e_m)_X| \leqslant \|x\|_X^2$. Then

$$\|Q - Q_n\|_{X \to Y} = \sup_{x \neq 0} \frac{\|Qx - Q_n x\|_Y}{\|x\|_X} \leqslant \Big(\sum_{m=n+1}^{\infty} \|Qe_m\|_Y \Big)^{1/2}.$$

Since the series $\sum_{m=1}^{\infty} \|Qe_m\|_Y^2$ converges by assumption, it follows in the usual way that $\lim_{n\to\infty} \sum_{m=n+1}^{\infty} \|Qe_m\|_Y = 0$ for the tail of the convergent series.

Finally, we obtain $\lim_{n\to\infty}\|Q-Q_n\|_{x\to Y}=0$ and the operator Q is compact by the Theorem of the subspace of compact operators. $\square$

Now, having Theorem 3, we can easily analyze the compactness of the operators that form the left-hand side of Equation (2.1).

Theorem 4. *The operators R, Γ^{-1}, B, and $\mathbb{K}$ acting from the space L^I to the space L^{II} are compact.*

Proof. It is almost trivial, using Theorem 3 and the preproved supports. Sections 3 and 4 are presented that the linear operators R, Γ^{-1}, B, and $\mathbb{K}$ are bounded in the pair of Hilbert spaces L^I and L^{II}. We have the basis $\left\{\sqrt{\frac{2}{\pi}}\frac{U_{m-1}(t)}{\sqrt{1+m^2}}\right\}_{m=1}^{\infty}$ for the space L^I. Also, the values $\left\|R\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^{II}}^2$, $\left\|\Gamma^{-1}\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^{II}}^2$, $\left\|B\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^{II}}^2$, and $\left\|\mathbb{K}\sqrt{\frac{2}{\pi}}\frac{U_{m-1}}{\sqrt{1+m^2}}\right\|_{L^{II}}^2$ are such that their series over all natural numbers m are convergent.

So, by Theorem 3, the operators R, Γ^{-1}, B, and $\mathbb{K}$ acting from the space L^I to the space L^{II} are compact. $\square$

Now note that the operator $hR+a\Gamma^{-1}+bB+\mathbb{K}$ is compact as a sum of the compact operators.

In [15] was presented the next theorem.

Theorem 5. *Let T be a compact operator and let I be the identity operator. The operators acting from a Banach space to Banach space. Then the following conditions are equivalent: a) the equation $(I-T)\,x=y$ is solvable for any right-hand side; b) the equation $(I-T)\,x=0$ does not have non-trivial solutions; c) the equation $(I-T)\,x=y$ is solvable in a unique way for any right-hand side.*

To prove the existence and uniqueness criterion for a solution of Equation (2.1), we also need the well-known statement on the compactness of the composition of compact and bounded operators, e.g., this statement is given in [15].

Now we present and prove the existence and uniqueness criterion for a solution of the second-kind complete hypersingular integral equation.

Criterion 1. *Second-kind complete hypersingular integral equation* (2.1) *has a unique solution in the space L^I for any right-hand side from the space L^{II} if and only if the corresponding homogeneous equation has no non-zero solution.*

Proof. Suppose that the homogeneous complete hypersingular integral equation of the second kind has no non-zero solution. Let us prove that a solution of Equation (3.4) exists and is unique. We transform hypersingular equation (3.4). Let us apply to both sides of Equation (3.4) the operator $-A^{-1}$ acting from the space L^{II} to the space L^I. By Theorem 2, this operator exists. We have

$$(Iu)\,(y)-\left(\left(A^{-1}\left(hR+a\Gamma^{-1}+bB+K\right)\right)u\right)(y)=-\left(A^{-1}f\right)(y)\,.$$

By remark to Theorem 4, the operator $hR + a\Gamma^{-1} + bB + \mathbb{K}$ acting from the space L^I to the space L^{II} is compact. By Theorem 2, the operator A^{-1} is bounded. Then, the operator $A^{-1}\left(hR + a\Gamma^{-1} + bB + \mathbb{K}\right)$ acting from the space L^{II} to the space L^I is compact too as the composition of compact and bounded operators. Since, by assumption, the homogeneous complete hypersingular integral equation does not have a non-zero solution, by Theorem 5 the equation is uniquely solvable for any right-hand side from the space L^{II}.

Suppose that a solution of Equation (3.4) exists and is unique. Then, by Theorem 5, we have that the corresponding homogeneous complete hypersingular integral equation of the second kind has not a non-zero solution. □

Let us note that Criterion 1 corresponds to a Fredholm's theorem. Also, note that the boundary second-kind complete hypersingular integral equation, obtained by solving the diffraction and scattering problems and presented in [17, 19], satisfies the criterion for the existence and uniqueness of a solution mentioned above. The corresponding homogeneous complete hypersingular integral equation does not have a non-zero solution. The basis of this is that the integral equation is obtained by equivalent transformations of a pair sum equation which is based on the Fourier series.

Let us justify the solvability of system of linear algebraic equations (5.4). Since the regularized operators are qualitatively identical to the original ones, we have that Criterion 1 naturally extend to regularized complete hypersingular integral equation of the second kind (4.8). Thus, we obtain criteria for the existence and uniqueness of a solution to equation (4.8) in the corresponding spaces. Suppose that a solution of complete hypersingular integral equation of the second kind (2.1) exists and is unique. Then, a solution of regularized complete hypersingular equation (4.8) also exists and is unique. As mentioned above, the interpolation polynomial constructed by the solution of system (5.4) is an exact solution of the system. If system of linear algebraic equations (5.4) is incompatible or has many solutions, then we get a contradiction to a solution of regularized equation (4.8) exists and is unique. A solution of the regularized equation is an interpolation polynomial, which is constructed uniquely. Finally, we have that if a solution of complete hypersingular integral equation (2.1) exists and is unique, then a solution of system of linear algebraic equations (5.4) also exists and is unique.

7 Rate of convergence of a sequence of approximate solutions to an exact solution

The estimate of the norm of the difference of an exact solution of Equation (2.1) and its approximate solution, i.e., the solution of Equation (4.8), is obtained by the following theorem from [7].

Theorem 6. *Let X and Y be the Banach spaces, let $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ be the subsequences of their finite-dimensional subspaces, respectively, let Q and Q_n be the linear operators acting from the space X to the space Y and from the spaces $\{X_n\}_{n=1}^{\infty}$ to the spaces $\{Y_n\}_{n=1}^{\infty}$, respectively, and let $Qx = y$ and*

$Q_n x_n = y_n$ be the equations. Assume that the following conditions are satisfied: a) the operator Q is invertible; b) the quantity $\varepsilon^{(n)} = \|Q - Q_n\|_{X_n \to Y} \xrightarrow[n\to\infty]{} 0$; c) for any number n we have $\dim X_n = \dim Y_n < \infty$; d) the quantity $\delta^{(n)} = \|y - y_n\|_Y \xrightarrow[n\to\infty]{} 0$.

Then, for all n satisfying the inequality $p_n = \left\|Q^{-1}\right\|_{Y\to X} \|Q - Q_n\|_{X_n \to Y} < 1$ the equation $Q_n x_n = y_n$ has a unique solution denoted by x_n^* for any right-hand side and there are two inequalities $\|x_n^*\|_{X_n} \leqslant \left\|Q^{-1}\right\|_{Y\to X_n} \|y_n\|_{Y_n}$ and $\left\|Q^{-1}\right\|_{Y\to X_n} \leqslant \frac{\|Q^{-1}\|_{Y\to X}}{1-p_n}$. The rate of convergence of the sequence of approximate solutions to exact solution denoted by x^* can be estimated as follows: $\frac{\alpha_n}{\|Q\|_{X\to Y}} \leqslant \|x^* - x_n^*\|_X \leqslant \alpha_n \left\|Q^{-1}\right\|_{Y\to X}$, where the number $\alpha_n = \|(y - y_n) + (Q_n - Q)\, x_n^*\|_Y$, and

$$\|x^* - x_n^*\|_X \leqslant \frac{\left\|Q^{-1}\right\|_{Y\to X}}{1 - p_n} \left(\|y - y_n\|_Y + p_n \|y\|_Y\right) = O\left(\varepsilon^{(n)} + \delta^{(n)}\right).$$

Thus, Theorem 6, estimates (4.2), (4.3), (4.4), (4.6), (4.7), and the parallelogram inequality

$$\begin{aligned}
&\left\|hR - hR_{n-2} + a\Gamma^{-1} - a\Gamma_{n-2}^{-1} + bB - bB_{n-2} + \mathbb{K} - \mathbb{K}_{n-2}\right\|_{\Pi^I \to L^{II}} \\
&+\|f_{n-2} - f\|_{L^{II}} \leqslant \frac{|h|}{\sqrt{n-2}} + \frac{\sqrt{2}}{2}\frac{|a|}{n-2} + \frac{\sqrt{3}}{2}\frac{|b|}{(n-2)^2} + \frac{3c_K}{(n-2)^{1+\alpha}} \\
&+\frac{2\sqrt{\pi}c_f}{(n-2)^\alpha} < \frac{\max\left\{|h|, \frac{\sqrt{2}}{2}|a|, \frac{\sqrt{3}}{2}|b|, 3c_K, 2\sqrt{\pi}c_f\right\}}{(n-2)^{\min\left\{\frac{1}{2},\alpha\right\}}}
\end{aligned}$$

are the basis for the main theorem of the section. Furthermore, this large inequality is a good point of view on the convergence of the presented numerical method. It is clearly seen that inequalities (4.2) and (4.7) determined the rate of convergence. Their right-hand sides are proportional to the fractions $\frac{1}{\sqrt{n-2}}$ and $\frac{1}{(n-2)^\alpha}$ respectively. The parameter α comes from belonging the function f to the set $C^{0,\alpha}_{[-1,1]}$. Quite often, in practice, the function f has large numerical indicators of smoothness. By the way, the estimation is improving. But inequality (4.2) keeps the rate of convergence at the level $\frac{1}{\sqrt{n-2}}$. Of course, if the parameter $|h|$ is equal to 0 and the right-hand side of Equation (2.1) has better smoothness, then the convergence rate is better, but that is another issue.

So, we have the main theorem on the convergence of the numerical method.

Theorem 7. *If the exact solution of complete hypersingular integral equation of the second kind* (2.1) *exists, for sufficiently large values of the number n, an approximate solution of Equation* (2.1) *is close to the exact solution and there is the following inequality:*

$$\|u - u_{n-2}\|_{L^I} \leqslant \frac{c}{(n-2)^{\min\left\{\frac{1}{2},\alpha\right\}}}, \tag{7.1}$$

where the number n is at least three, the constant c is n-independent and equal to the maximum of the numbers $|h|$, $\frac{\sqrt{2}}{2}|a|$, $\frac{\sqrt{3}}{2}|b|$, $3c_K$, and $2\sqrt{\pi}c_f$ from estimates (4.2), (4.3), (4.4), (4.6), *and* (4.7).

Let us remark that inequality (7.1) is an important point for the analysis of applied problems. It helps to estimate the rate of convergence of the physical sense linear functionals of an approximate solution to their values of an exact solution. For example, we have to calculate such functionals for solving the diffraction and scattering problems which are reduced to the considered complete hypersingular integral equation of the second kind.

8 Model problem and numerical convergence

In this section, we illustrate the presented numerical method. We use it to solve a model problem. The model problem is the complete hypersingular integral equation of the second kind and its exact solution. The model problem was constructed using the methods of exact calculations of singular integrals were shown in [25] and the relationship between singular and hypersingular integrals was presented in [9]. We will obtain the subsequence of approximate solutions and compare them with an exact solution. This allows us to estimate the numerical convergence of our method and characterize it empirically.

The model hypersingular integral equation includes the first four terms of the left-hand side of Equation (2.1). Three of them were regularized and have influenced the rate of convergence of numerical method with estimates (4.2)–(4.4). So, the model complete hypersingular integral equation of the second kind has the following form:

$$\begin{aligned} &hu\,(y)\sqrt{1-y^2}-\frac{1}{\pi}\int_{-1}^{1}\frac{u\,(t)}{(t-y)^2}\sqrt{1-t^2}dt+\frac{a}{\pi}\int_{-1}^{1}\frac{u\,(t)}{t-y}\sqrt{1-t^2}dt \\ &+\frac{b}{\pi}\int_{-1}^{1}\ln|t-y|\,u\,(t)\sqrt{1-t^2}dt=h\sin y\,\mathrm{sh}\sqrt{1-y^2}+\sin y\,\mathrm{ch}\sqrt{1-y^2} \\ &+\frac{y}{\sqrt{1-y^2}}\cos y\,\mathrm{sh}\sqrt{1-y^2}+a\left(\cos y\,\mathrm{ch}\sqrt{1-y^2}-1\right) \\ &+b\int_0^y\left(\cos\tau\,\mathrm{ch}\sqrt{1-\tau^2}-1\right)d\tau. \end{aligned} \tag{8.1}$$

The exact solution of integral equation (8.1) has the next form: $u\,(t)=-\frac{\sin t\,\mathrm{sh}\sqrt{1-t^2}}{\sqrt{1-t^2}}$. Denote by u_{n-2} an approximate solution of Equation (8.1). Note that the function u is uneven and real. Then, it is natural to expect that the function u_{n-2} is uneven and real too.

Let us remark that Equation (8.1) is close in form to the equations of the mathematical theory of diffraction and scattering of waves presented in [17,19]. Comparison of the exact solution of Equation (8.1) with the approximate solutions obtained using the numerical method is an empirical validation test and a characteristic of the method applicability. As noted above, the regularization of operators influences the rate of convergence and the comparison results numerically estimate this effect.

Table 1 shows the results of a numerical analysis of model complete hypersingular integral equation of second kind (8.1) by the numerical method.

For the numerical analysis we suppose that the parameters $h=0.1-0.3i$, $a=2i$, and $b=8$. Such parameter values are associated with applied problems

Table 1. The values of exact and approximate solutions.

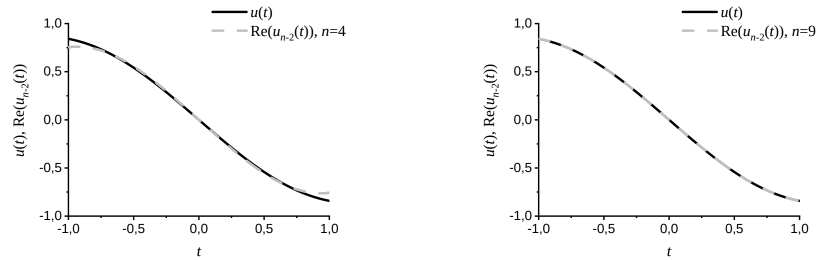
n	j	Exact solution, $u\left(t_{0j}^n\right)$	Real part of approximate solution, $\mathrm{Re}\left(u_{n-2}\left(t_{0j}^n\right)\right)$	Imaginary part of approximate solution, $\mathrm{Im}\left(u_{n-2}\left(t_{0j}^n\right)\right)$	Modulus of difference between exact and approximate solutions, $\left\|(u-u_{n-2})\left(t_{0j}^n\right)\right\|$
4	1	−0.766001565264209	$-\underline{0.7}44006372879521$	2.20×10^{-2}	2.20×10^{-2}
	2	−0.352087876852657	$-\underline{0.3}64834348544664$	-6.21×10^{-3}	1.27×10^{-2}
9	1	−0.827047091227887	$-\underline{0.82704}5749330590$	-2.33×10^{-7}	1.34×10^{-6}
	2	−0.766001565264209	$-\underline{0.76600}2675648515$	9.98×10^{-8}	1.11×10^{-6}
	3	−0.617019617229588	$-\underline{0.61701}8843519980$	-1.18×10^{-8}	7.74×10^{-7}
	4	−0.352087876852657	$-\underline{0.35208}8266848951$	-7.62×10^{-8}	3.90×10^{-7}
15	1	−0.836215500091829	$-\underline{0.83621550009}2045$	2.25×10^{-14}	2.16×10^{-13}
	2	−0.817565113714310	$-\underline{0.817565113714}107$	-1.77×10^{-14}	2.02×10^{-13}
	3	−0.777525770694384	$-\underline{0.777525770694}565$	1.12×10^{-14}	1.82×10^{-13}
	4	−0.705142985386376	$-\underline{0.705142985386}223$	-4.65×10^{-15}	1.53×10^{-13}
	5	−0.590335631293662	$-\underline{0.590335631293}782$	-1.21×10^{-15}	1.19×10^{-13}
	6	−0.428845903728567	$-\underline{0.428845903728}486$	5.84×10^{-15}	8.15×10^{-14}
	7	−0.226464167898581	$-\underline{0.226464167898}622$	-8.28×10^{-15}	4.12×10^{-14}

and do not reduce the contribution of the regularized operators. Since the functions u and u_{n-2} are uneven and the identity $t_{0n-j}^n = -t_{0j}^n$ for $j = \overline{1,n}$, the j-index value runs over all integers from 1 to the integer part of the ratio $\frac{n}{2}$. In numbers presented in Table 1, the significant digits are underlined.

Let us estimate the numerical convergence of the method. The data listed in Table 1 show that an increase in the number n leads to a better approximation of the exact solution by approximate solutions. So, when the number $n = 4$, then we have a difference between exact and approximate solutions starting from the second digit after a decimal sign. When the number $n = 9$, then the difference is starting from the sixth digit. And when the number $n = 15$, we have the difference starting from the twelfth and thirteenth digits. Since the graphic accuracy is achieved when an approximate solution has three or four significant digits, we have very good numerical convergence. Note that the numerical convergence significantly exceeds a priori estimate (7.1). This is a positive point of the method.

The approximate solutions have a non-trivial imaginary part. This is because parameters h and a are complex numbers. But the imaginary part of an approximate solution is expected close to zero. A real part is close to the exact solution. The modulus of the difference between the exact and approximate solutions is small and increases slightly as the argument is approaching an edge of interval $(-1, 1)$. Figure 1 a) shows the graphs of the functions u and u_{n-2} when the number $n = 4$. We see that the functions are very close. Although as shown in Table 1, the approximate solution has just two significant digits. Some differences in the graphs appear near the edges of interval $(-1, 1)$.

For analyzing the numerical convergence and error of the method, the obtained data were averaged. We used the following rules:



a) b)

Figure 1. The graphs of exact and approximate solutions.

- $E_{n-2} = \int_{-1}^{1} |u(t) - u_{n-2}(t)|\, dt$, the sample mean for estimating of error;

- $\sqrt{D_{n-2}} = \sqrt{\int_{-1}^{1} \left(|u(t) - u_{n-2}(t)|\, dt - E_{n-2}\right)^2 dt}$, the square root of the unbiased sample variance for estimating the mean deviation;

- $M_{n-2} = \max_{-1 \le t \le 1} |u(t) - u_{n-2}(t)|$, the maximum distance between the exact and approximate solutions for estimating the maximum deviation;

- $\|u - u_{n-2}\|_{L^I}$, the norm of the difference between exact and approximate solutions for estimating the deviation in the space L^I;

- $\|u - u_{n-2}\|_{L^{II}}$, for estimating the deviation in the space L^{II}.

Table 2 presents the averaged data about error analysis of the numerical method. There are the mean error, the mean deviation, the maximum deviation, and deviations in the spaces L^I and L^{II}. These results were obtained by using the numerical method for integral equation (8.1) with the above parameters. We see that an increase in the number n leads to a quick decrease of the mean error from 4.37×10^{-2} when the number $n = 4$ to 2.35×10^{-13} when the number $n = 15$. The other error characteristics are quickly decreasing too. Let us remark that this exceeds the a priori estimate of the method and characterizes the numerical convergence as good.

Figure 1 b) presents the graphs of the functions u and u_{n-2} when the number $n = 9$. We see that the functions graphically coincide. The approximate solution has six significant digits, as shown in Table 2. Thus, changing the number n from 4 to 9 leads to the graphic coincidence of the approximate and exact solutions on all points of interval $(-1, 1)$.

As presented in Table 1 and Table 2, when the number $n = 15$, the functions u and u_{n-2} are closer. The values of function u_{n-2} have twelfth and thirteenth significant digits.

The presented numerical analysis results of model second-kind hypersingular integral equation (7.1) by the developed method are shown a slight difference between the exact and approximate solutions in the neighbourhood of the edges

Table 2. The averaged error analyze data.

n	Mean error, E_{n-2}	Mean deviation, $\sqrt{D_{n-2}}$	Maximum deviation, M_{n-2}	Deviation in the space L^I, $\|u-u_{n-2}\|_{L^I}$	Deviation in the space L^{II}, $\|u-u_{n-2}\|_{L^{II}}$
3	1.84×10^{-1}	1.85×10^{-1}	3.34×10^{-1}	1.51	6.79×10^{-1}
4	4.37×10^{-2}	4.15×10^{-2}	9.24×10^{-2}	3.27×10^{-1}	1.23×10^{-1}
5	7.94×10^{-3}	8.75×10^{-3}	2.37×10^{-2}	1.01×10^{-1}	2.66×10^{-2}
6	7.88×10^{-4}	7.95×10^{-4}	2.44×10^{-3}	2.16×10^{-3}	5.12×10^{-3}
7	1.33×10^{-4}	1.57×10^{-4}	5.73×10^{-4}	2.60×10^{-3}	4.75×10^{-4}
8	9.27×10^{-6}	9.46×10^{-6}	3.62×10^{-5}	5.30×10^{-4}	8.58×10^{-5}
9	1.40×10^{-6}	1.74×10^{-6}	7.86×10^{-6}	4.06×10^{-5}	5.54×10^{-6}
10	7.54×10^{-8}	7.82×10^{-8}	3.53×10^{-7}	7.42×10^{-6}	9.13×10^{-7}
11	1.05×10^{-8}	1.32×10^{-8}	6.93×10^{-8}	4.16×10^{-7}	4.53×10^{-8}
12	4.44×10^{-10}	4.71×10^{-10}	2.42×10^{-9}	6.77×10^{-8}	6.72×10^{-9}
13	5.56×10^{-11}	7.27×10^{-11}	4.30×10^{-10}	2.99×10^{-9}	2.70×10^{-10}
14	2.01×10^{-12}	2.15×10^{-12}	1.22×10^{-11}	4.35×10^{-10}	3.62×10^{-11}
15	2.35×10^{-13}	3.04×10^{-13}	1.98×10^{-12}	1.59×10^{-11}	1.22×10^{-12}

of interval $(-1,1)$ when the number n is small. In the central part of the interval, accuracy is increasing. The number of significant digits of the real part of the approximate solution increases and the imaginary part tends to zero. When the number n increases the mean error and deviations are quickly decreasing. The deviation is much smaller than a priori estimate (7.1). When the number $n=9$, the graphs of exact and approximate solutions graphically coincide as shown in Figure 1 b). Finally, the obtained numerical results marked the numerical convergence of our method as good.

9 Conclusions

We presented the new Nyström-type numerical method for solving the complete hypersingular integral equation of the second kind.

The main advantage of the numerical method lies in the content of the matrix of the system of linear algebraic equations for the values of the unknown function at the interpolation nodes. All elements of the main diagonal of the matrix are proportional to the number n. The other matrix elements are inversely proportional to the number n or limited. Thus, we have a well-conditioned matrix and it looks like diagonal dominance one. Moreover, in the numerical analysis of diffraction and scattering problems, we always have the experimentally substantiated diagonal dominance of the matrices.

The numerical method is justified. The criterion of existence and uniqueness is proved. The criterion has related the existence and uniqueness of a solution of the hypersingular equation by a solution of the corresponding homogeneous equation. It is convenient to use the criterion in solving the diffraction and scattering problems.

The estimate of the norm of the difference between exact and approximate solutions is obtained. It is inversely proportional to the number not exceeding $\sqrt{n-2}$. However, the numerical convergence exceeds this estimate. The numerical analysis results of the model problem are shown. These data are characterized the numerical method as good enough.

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