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Uniqueness of Degenerating Solutions to a Diffusion-Precipitation Model for Clogging Porous Media

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Abstract. The current article presents a degenerating diffusion-precipitation model including vanishing porosity and focuses primarily on uniqueness results. This is accomplished by assuming sufficient conditions under which the uniqueness of weak solutions can be established. Moreover, a proof of existence based on a compactness argument yields rather regular solutions, satisfying these unique conditions. The results show that every strong solution is unique, though a slightly different condition is additionally required in three dimensions. The analysis presents particular challenges due to the nonlinear structure of the underlying problem and the necessity to work with appropriate weights and manage possible degeneration.

Keywords: evolving porous media, degenerate equations, clogging, weighted spaces, uniqueness.

AMS Subject Classification: 35D30; 35K65; 76R50.

1 Introduction

The research of transport in porous media is motivated by a variety of applications. Recently, there has been a surge of interest in the study of an evolving porous matrix caused by various reactions, such as crystal precipitation or biofilm growth. Such precipitants or biofilms attach to the surface of the solid matrix, occupying pore space. Thus, these attachments cause geometrical changes in the microstructure, which significantly impact the hydrodynamic properties of the porous medium and hence impede the mass transport of dissolved substances within the pores. Substrate transport, in turn, strongly impacts interface reactions, altering the microstructure. Modeling such processes generally results in a strongly coupled system of nonlinear partial differential equations (PDE). From a practical point of view, e.g. for computational feasibility, an upscaled (averaged) model at the macro-scale is of major importance compared to a pore-scale model. [12] introduced an extension of formal two-scale asymptotic expansion to a level set framework capable of capturing the evolving solid-liquid interface. This method has recently been applied to crystal precipitation [10, 12], locally-periodic porous media [13], drug delivery systems [5], non-isothermal media [2], and biofilm growth [8, 9].

However, little attention has been paid to the investigation of degenerate transport equations in evolving porous media due to clogging effects, which are rarely studied analytically. Clogging phenomena in porous media, on the other hand, appear and thus are of particular interest. In [1], degeneracies arising in linear elliptic equations describing two-phase mixtures, such as partially melted materials, were managed by scaling the unknowns appropriately. Additionally, a stabilized variational formulation was used to show existence and uniqueness of a solution over the entire domain. In [6], the porous medium's porosity $\theta : \Omega \times (0, T) \to [0, 1)$ was assumed to be a given function, where the degenerate case $\theta(x, t) = 0$ was of particular interest and thus explicitly admissible. The degeneracy was handled and analytical results were obtained by introducing appropriate weighted function spaces and including the degenerate parameters as weights. Specifically, the non-vanishing parts of the hydrodynamic parameters were proposed to belong to the Muckenhoupt class A_2.

The present study considers a model of [7, 10] that describes the diffusive transport of a reactive substance in a saturated porous medium, including variable porosity described by a system of coupled (partial) differential equations. In [10], a two-scale asymptotic expansion in a level set framework was used to derive an effective, nonlinear diffusion equation coupled to an ordinary differential equation (ODE) for porosity change. This modeled system of PDEs was also analyzed in [10], though clogging effects were excluded. Recently, the analysis of degenerating equations due to vanishing porosity was included in [7]. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an open and bounded domain. In the current article, we reconsider this diffusive transport equation, cf. [7, 10]:

$$\begin{aligned} \theta\partial_t c - \nabla \cdot \mathbb{D}(\theta)\nabla c &= (\tfrac{c}{\rho} - 1) f(c, \theta) && \text{in } \Omega \times (0, T), \\ c &= 0 && \text{on } \partial\Omega \times (0, T), \\ c(\,.\,, 0) &= c_0 && \text{in } \Omega \end{aligned} \tag{1.1a}$$

with the substrate concentration c, porous medium's porosity $\theta \in [0, 1)$, constant density $\rho > 0$ of the precipitation, the reaction rate f, and the effective diffusivity $\mathbb{D}$ (depending on θ). Here, the homogeneous boundary condition and the initial data c_0 are assumed. The precipitation reaction rate is given by $f(c, \theta) := \theta^\sigma c$ with $\sigma \geq 2$. In contrast to the models considered in [8, 10, 12], where for a reasonable shape of the microstructure the specific grain's surface behaves like $\theta^{\frac{1}{2}}$, we assume a larger exponent and hence a faster decay of the reaction rate with respect to the porosity for analytical feasibility.

We assert that the microstructure's geometry is parametrized by a single parameter, which is represented by the porosity. In this case, the hyperbolic level set equation (which generally describes the evolving microstructure) reduces to an ordinary differential equation describing the change of this parameter, cf. [10, 12]. In the current underlying situation, such an assumption is reasonable since the reaction rate f causing the microstructure change does not depend on microscopic values, i.e., it occurs uniformly along the fluid-solid interface. Therefore, the evolution of the porosity θ caused by precipitation is given via the following ODE:

$$\begin{aligned} &\partial_t\theta = -\tfrac{1}{\rho} f(c,\theta) \quad \text{in } \Omega\times(0,T), \\ &\theta(\,.\,,0) = \theta_0 \qquad\qquad\qquad\qquad \text{in } \Omega\,. \end{aligned} \tag{1.1b}$$

In addition to the changeable properties of the porous medium in principle, the possibility of degenerating transport equations due to clogging complicates the analytical study of the present model (1.1) significantly.

Although, it is generally difficult to characterize, the effective diffusivity $\mathbb{D}$ is essential for modeling the substrate transport in porous media. Consequently, porosity-diffusivity models such as linear relations or power-laws are typically used, cf. [4]. It is often assumed that diffusivity satisfies the constitutive law $\mathbb{D}(\theta) = \alpha\theta^d$. For instance, Penman (1940) suggests $d = 1$ and $\alpha = 0.66$, while for $\alpha = 1$, Marshall (1959) suggests $d = \frac{3}{2}$, and Millington (1959) $d = \frac{4}{3}$. However, the case $d < 1$ is not of interest for applications since there exist the following analytically derived bounds: the Voigt-Reiss bound $\mathbb{D}(\theta) \lesssim \theta$ and the n-dimensional Hashin-Shtrikman bound $\mathbb{D}(\theta) \lesssim \frac{n-1}{n-\theta}\theta$ for the effective diffusion, cf. [4]. In particular, for small porosities, the three-dimensional upper Hashin-Shtrikman bound $\frac{2}{3-\theta}\theta$ is approximated linearly by $\frac{2}{3}\theta \approx 0.66\theta$, which is exactly the relation proposed by Penman. Hence, the specific choice of $d = 1$ provides a reasonable relation for small porosities. This is exactly the focus of the present research since, at the limit of clogging, the porosity θ vanishes. Thus the diffusivity $\mathbb{D} : [0, 1) \to [0, \infty)$ is assumed to be a scalar-valued, monotonously increasing map depending on the porosity θ. For readability, it is reasonable to assume that the diffusivity satisfies $\mathbb{D}(\theta) = \theta^d$ for some $d \geq 1$, i.e., $\alpha = 1$, cf. [7].

In [7], the author has already analyzed the coupled degenerating PDE–ODE model (1.1). An adjusted Rothe method was used to verify the existence of weak solutions in weighted function spaces. In particular, the decay behavior of the concentration c with respect to the porosity θ was studied. However, in contrast to [10], the model (1.1) was solved in [7] even for substantially degenerating hydrodynamic parameters. In this respect, the obtained results in [7] extended the knowledge of the model introduced in [10] and actually allowed the investigation of clogging processes.

Nevertheless, proof of uniqueness of weak solutions to (1.1) was missing in [7] since it can not be done by a simple argument. Due to nonlinearity and degenerating θ-weights in (1.1a), a more skillful approach is necessary. In this article, the analysis of the degenerating diffusion-precipitation model (1.1) was continued and sufficient conditions under which uniqueness of weak solutions can be established were formulated. Moreover, another proof of existence, which is based on a compactness argument, yields rather regular solutions to (1.1), satisfying these uniqueness conditions in the two-dimensional case.

As a result, in two dimensions every strong solution is unique. In contrast, three-dimensional solutions satisfy these sufficient conditions at least partially, however, as a consequence it is open whether uniqueness holds.

The particular challenges of the following analysis arise due to the nonlinear structure of the underlying problem and the necessity to work with appropriate θ-weights and manage possible degeneration.

This article is organized as follows: Section 2 summarizes the main results and discusses the central topic. The more comprehensive proofs of the uniqueness (Theorem 2) and regularity properties (Lemma 2) are discussed in Sections 3 and 4, respectively. Theorem 3 establishes the existence of solutions to (1.1). Finally, this article completes with a brief conclusion. The norm of the Banach space L^p, $p \in [1,\infty]$, is denoted by $\|\,.\,\|_p$ throughout the paper. All subsequent norms are intuitively denoted, e.g. $\|u\|_{L^\infty(L^p)} = \sup_{t\in(0,T)} \|u(t)\|_p$ for $u \in L^\infty(0,T;L^p(\Omega))$, $p \geq 1$. Furthermore, C describes positive constants, where the value may differ from one occasion to another. Furthermore, the notation $a \lesssim b$ for real numbers $a, b \in \mathbb{R}$ denotes that $a \leq Cb$ for some constant $C > 0$.

2 Main results

To investigate the transport equation (1.1a) for clogging scenarios, it is reasonable to define an appropriate θ-weighted linear space for given θ:

$$V_\theta(\Omega) := \{u \in L^2(\Omega) : \mathbb{D}(\theta)^{\frac{1}{2}} \nabla u \in (L^2(\Omega))^n \text{ and } u = 0 \text{ on } \partial\Omega\}$$

with the corresponding inner product

$$(u, v)_{V_\theta} := (u, v)_2 + (\mathbb{D}(\theta)^{\frac{1}{2}} \nabla u\,,\, \mathbb{D}(\theta)^{\frac{1}{2}} \nabla v)_2\ .$$

Here $(\,.\,,\,.\,)_2$ denotes the inner product of $L^2(\Omega)$. If the gradient of the unknown porosity $\theta : \Omega_T \to [0,1)$ with $\Omega_T = \Omega \times (0,T)$ satisfies

$$\theta^{\frac{d}{2}} \in L^\infty(0,T;H^1(\Omega))\ , \tag{2.1}$$

the linear space $V_\theta(\Omega)$ equipped with the above inner product is a Hilbert space, cf. [1, 7]. Let us further introduce

$$\mathcal{X}_\theta := \{L^2(0,T;V_\theta(\Omega)) \,|\, \theta\partial_t c \in L^2(0,T;(V_\theta(\Omega))^*)\} \text{ and } \mathcal{Y} := H^1(0,T;L^2(\Omega)),$$

where $(V_\theta(\Omega))^*$ denotes the dual space of $V_\theta(\Omega)$. In general there hold the following inclusions for a bounded porosity weight $\theta : \Omega_T \to [0,1)$:

$$H_0^1(\Omega) \subset V_\theta(\Omega) \subset L^2(\Omega) \subset (V_\theta(\Omega))^* \subset H^{-1}(\Omega)\ ,$$

i.e., $L^2(0,T;V_\theta(\Omega))$ is interpreted as a subspace of the rigid Bochner space $L^2(0,T;L^2(\Omega))$. With respect to this θ-weighted function space we define weak solvability of our underlying diffusion-precipitation model (1.1) with homogeneous Dirichlet boundary condition, diffusion parameter $\mathbb{D}(\theta) = \theta^d$ and reaction rate $f(c,\theta) = \theta^\sigma c$ as follows:

DEFINITION 1. A pair $(c,\theta) \in \mathcal{X}_\theta \times \mathcal{Y}$ is called a *weak solution* to the coupled system (1.1) if for all test functions $(\varphi_1,\varphi_2) \in H_0^1(\Omega) \times L^2(\Omega)$ and a. e. $t \in (0,T)$ there holds

$$\langle \theta\partial_t c\,,\,\varphi_1\rangle_{V_\theta^*,V_\theta} = -\int_\Omega \mathbb{D}(\theta)\nabla c\nabla\varphi_1 + \int_\Omega \Big(\frac{c}{\rho}-1\Big) f(c,\theta)\varphi_1\,, \tag{2.2a}$$

$$\int_\Omega (\partial_t\theta)\varphi_2 = -\frac{1}{\rho}\int_\Omega f(c,\theta)\varphi_2, \tag{2.2b}$$

and if (c,θ) takes the initial value $(c_0,\theta_0) \in L^2(\Omega)^2$ in the sense

$$|(c(t)-c_0,\phi)_2| + |(\theta(t)-\theta_0,\phi)_2| \xrightarrow{t\to 0} 0$$

for all $\phi \in L^2(\Omega)$.

This weak formulation was already considered in [7]. There it was of particular interest whether the substrate concentration c remains bounded within clogged regions. Depending on the density ρ of the precipitant, solution c may even vanish with respect to θ. More precisely, for some parameter $p>0$, the concentration c decays at least as θ^p. In [7] the following existence result was shown by an adjusted Rothe method:

Theorem 1. *Let the parameters $\sigma \geq 1$, $p \geq 0$ and $d \in [1, 1+2p]$ satisfy $\frac{d-3}{2} \leq \sigma + p$. Moreover, the nonnegative initial data $c_0, \theta_0 \in L^\infty(\Omega)$ are assumed to fulfill*

$$\|c_0\|_\infty \leq \tfrac{\rho}{1+p},\;\; \big\|\theta_0^{-p}c_0\big\|_2 < \infty,\;\; \theta_0 \in [0,1) \;\text{ and }\; \Big\|\theta_0^{-1-\frac{2p+1-d}{2}}\nabla\theta_0\Big\|_2 < \infty.$$

Then for all $T>0$ there exists a nonnegative solution $(c,\theta) \in \mathcal{X}_\theta \times \mathcal{Y}$ to (2.2) with

$$\sup_{t\in(0,T)} \|c(t)\|_\infty \leq \|c_0\|_\infty \quad \text{and} \quad \sup_{t\in(0,T)} \|\theta(t)\|_\infty \leq \|\theta_0\|_\infty\ .$$

In particular, in the L^2-norm the solution c and the gradient of θ are decaying of order p and $1+\frac{2p+1-d}{2}$, respectively, with respect to θ.

Proof. See [7, Theor. 4.2]. □

It should be noted that the preceding terms are well-defined, although $\theta_0 = 0$ is explicitly allowed. The reciprocal initial data θ_0^{-1} never appears on its own, but always in combination with a multiplicative partner compensating for the bad behavior of this weight.

This theorem reads less technical if only boundedness of the solution c is aimed at and the decay property is neglected, i.e., $p=0$. In such a situation the behavior of c in the region of vanishing porosity $\Omega_\theta(t) = \{x \in \Omega : \theta(x,t) = 0\}$ is not clear. Therefore, we assume at least slight decay, ensuring that besides θ, c also vanishes in this region.

In [7], the difficulty of proving uniqueness of weak solutions was already mentioned. In fact, this is beside the strong nonlinear structure of (2.2) with respect to (c,θ) also due to degenerating θ-weights. In contrast to the non-clogging case [10] (i.e. $\theta \geq \delta > 0$), in the present situation of vanishing porosity, we need to control adequately θ-weighted norms.

Besides a stronger restriction on the parameter σ, the conditions

$$\|\theta\nabla c\|_{L^2(L^{2n})} < \infty \quad \text{and} \quad \left\|\theta^{-1}\nabla\theta\right\|_{L^\infty(L^{2n})} < \infty \tag{2.3a}$$

and additionally

$$\left\|\theta^{\frac{d-3}{2}}\nabla\theta\right\|_{L^{\frac{6+\kappa}{3+\kappa}}(L^{2n+\kappa})} < \infty \quad \text{for some } \kappa > 0 \tag{2.3b}$$

entail uniqueness in the following main result.

Theorem 2 [Uniqueness]. *Let $\sigma \geq 2$, $d \geq 1$ and $(c,\theta) \in \mathcal{X}_\theta \times \mathcal{Y}$ be a bounded and nonnegative weak solution to* (2.2) *fulfilling* (2.3) *such that c vanishes in the clogged region Ω_θ and $\theta(x,t) \in [0,1)$ for a.e. $(x,t) \in \Omega_T$. Then (c,θ) coincides with any weak solution also satisfying the property* (2.3a).

Proof. See Section 3. □

In comparison to (2.3a), the condition (2.3b) includes a weaker θ-weight with an exponent $\frac{d-3}{2} \geq -1$ and requires less time integrability. However, it is a slightly stronger assumption with respect to spatial integrability. Besides, condition (2.3a) involves (2.1) such that V_θ is, in fact, a Hilbert space.

To establish solutions to (2.2) that satisfy the conditions (2.3), more regularity of such solutions is required. The proof of Theorem 1 was based on an adjusted Rothe method, and showing regularity rigorously with the discretization technique of the Rothe method would be unnecessarily technical. Therefore, in contrast to [7], the present study used a more elegant compactness argument yielding the existence of rather regular solutions. Thus, for $\varepsilon > 0$ we consider the non-degenerating modification of (1.1)

$$\begin{aligned}
&\theta_\varepsilon\partial_t c_\varepsilon - \nabla\cdot\mathbb{D}(\theta_\varepsilon)\nabla c_\varepsilon = (c_\varepsilon/\rho - 1)f_\varepsilon(c_\varepsilon,\theta_\varepsilon) \quad \text{in } \Omega\times(0,T),\\
&c_\varepsilon = 0 \quad \text{on } \partial\Omega\times(0,T),\\
&c_\varepsilon(\,.\,,0) = c_0 \quad \text{in } \Omega,\\
&\partial_t\theta_\varepsilon = -f_\varepsilon(c_\varepsilon,\theta_\varepsilon)/rho \quad \text{in } \Omega\times(0,T),\\
&\theta_\varepsilon(\,.\,,0) = \theta_{0,\varepsilon} \quad \text{in } \Omega.
\end{aligned} \tag{2.4}$$

As demonstrated below, these solutions also satisfy (2.3a) (cf. Lemma 2) and thus give this property to the limit satisfying (1.1). Theorem 2 then implies the equality of this limit to any strong solution satisfying (2.3b) or any weak solution satisfying (2.3). However, since the limit solves a degenerating system, it is necessary to derive uniformly bounded estimates in appropriate θ-weighted norms.

In (2.4), the initial data θ_0, as well as the right-hand side f, are replaced by $\theta_{0,\varepsilon} \in H^2(\Omega)$ with $\theta_{0,\varepsilon}(x,t) \in [\varepsilon,1)$ for a.e. $(x,t) \in \Omega_T$ and

$$f_\varepsilon(c_\varepsilon,\theta_\varepsilon) := \begin{cases} f(c_\varepsilon,\theta_\varepsilon) & \text{for } \theta_\varepsilon > \varepsilon,\\ 0 & \text{else,}\end{cases}$$

respectively, avoiding the porosity from falling below $\varepsilon > 0$. According to the existence theorem of Caratheodory, there exists a unique, absolutely continuous solution $\theta_\varepsilon \in \mathcal{Y}$. Obviously, testing the evolution equation of the porosity in (2.4) with $\min\{\theta_\varepsilon - \varepsilon, 0\}$ implies that this system does not degenerate since $\theta_\varepsilon \geq \varepsilon$ remains valid during the evolution. Thus, standard parabolic theory [3, Chap. III] can be applied with respect to the usual function space

$$\mathcal{X} := L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \ . \tag{2.5}$$

In this case of non-vanishing porosity $\theta_\varepsilon \geq \varepsilon > 0$, the Sobolev space $H_0^1(\Omega)$ is isomorph to $V_{\theta_\varepsilon}(\Omega)$ and Definition 1 coincides with the typical definition of weak solvability with respect to (2.5). Therefore, for each $\varepsilon > 0$, we obtain a unique solution $(c_\varepsilon, \theta_\varepsilon) \in \mathcal{X} \times \mathcal{Y}$, cf. [8]. Nevertheless, the norms of c_ε in $\mathcal{X}$ would blow up as $\varepsilon \to 0$. For this reason, uniformly bounded estimates in θ-weighted norms are needed. The following results concern additional regularity properties that are satisfied by $(c_\varepsilon, \theta_\varepsilon)_{\varepsilon>0}$ in such norms and that are actually given to the limit (c, θ) solving (1.1).

Let $c_0 \in L^\infty(\Omega)$ such that $\|c_0\|_\infty \leq \rho$ and $\theta_{0,\varepsilon}(x) \in [\varepsilon, 1)$ for a.e. $x \in \Omega$ and $\varepsilon > 0$ be nonnegative initial data. Then, similar to [7, Theorem 2.3], the solution $(c_\varepsilon, \theta_\varepsilon) \in \mathcal{X} \times \mathcal{Y}$ to (2.4) is nonnegative and bounded:

$$\sup_{t\in(0,T)} \|c_\varepsilon(t)\|_\infty \leq \|c_0\|_\infty \quad \text{and} \quad \theta_\varepsilon(x,t) \in [\varepsilon, \theta_{0,\varepsilon}(x)] \tag{2.6}$$

for a.e. $(x,t) \in \Omega_T$. If sufficient regularity on the data is additionally assumed, i.e., $c_0 \in H_0^1(\Omega)$, the unique weak solution c_ε even belongs to

$$c_\varepsilon \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \ , \tag{2.7}$$

cf. [3, Chap. III]. However, since we are interested in the limiting solution of (1.1) for $\varepsilon \to 0$, regularity results with respect to appropriate θ-weighted function spaces are needed, such that corresponding norms are uniformly bounded in ε.

Lemma 1. *Let the initial data be given as above with $\varepsilon > 0$, $\|c_0\|_\infty \leq \rho$, $\theta_{0,\varepsilon}(x) \in [\varepsilon, 1)$ for a.e. $x \in \Omega$ and $\mathbb{D}(\theta_{0,\varepsilon})^{\frac{1}{2}}\nabla c_0 \in L^2(\Omega)$. Then the solution c_ε fulfills also the following properties:*

$$\theta_\varepsilon^{\frac{d}{2}}\nabla c_\varepsilon \in L^\infty(0,T;L^2(\Omega)) \qquad \textit{and} \qquad \theta_\varepsilon^{\frac{1}{2}}\partial_t c_\varepsilon \in L^2(0,T;L^2(\Omega)) \ . \tag{2.8}$$

Moreover, the corresponding norms are uniformly bounded with respect to $\varepsilon > 0$.

Proof. In fact, the assertion (2.8) follows directly from (2.7) and the boundedness of θ_ε. However, the uniform boundedness of the norms is the principal point of this Lemma. Testing $(2.4)_1$ with $\partial_t c_\varepsilon$ leads to

$$\int_0^t (\theta_\varepsilon \partial_t c_\varepsilon, \partial_t c_\varepsilon)_2 = \int_0^t \left\|\theta_\varepsilon^{\frac{1}{2}}\partial_t c_\varepsilon\right\|_2^2 ,$$
$$\int_0^t \int_\Omega \mathbb{D}(\theta_\varepsilon)\nabla c_\varepsilon \nabla(\partial_t c_\varepsilon) = -\tfrac{1}{2}\int_0^t \int_\Omega \partial_t(\mathbb{D}(\theta_\varepsilon))|\nabla c_\varepsilon|^2 + \tfrac{1}{2}\Big[\int_\Omega \mathbb{D}(\theta_\varepsilon)|\nabla c_\varepsilon|^2(\tau)\Big]_{\tau=0}^t$$

$$= -\frac{d}{2}\int_0^t\int_\Omega \theta_\varepsilon^{d-1}\partial_t\theta_\varepsilon|\nabla c_\varepsilon|^2 + \tfrac{1}{2}\left\|\mathbb{D}(\theta_\varepsilon)^{\frac{1}{2}}\nabla c_\varepsilon(t)\right\|_2^2 - \tfrac{1}{2}\left\|\mathbb{D}(\theta_{0,\varepsilon})^{\frac{1}{2}}\nabla c_0\right\|_2^2$$

and further by (2.6) and Young's inequality

$$\int_0^t\int_\Omega (\tfrac{c_\varepsilon}{\rho}-1)f_\varepsilon(c_\varepsilon,\theta_\varepsilon)\partial_t c_\varepsilon \le \tfrac{1}{2}\int_0^t \left\|\theta_\varepsilon^{\sigma-\frac{1}{2}}c_\varepsilon\right\|_2^2 + \tfrac{1}{2}\int_0^t\left\|\theta_\varepsilon^{\frac{1}{2}}\partial_t c_\varepsilon\right\|_2^2 . \tag{2.9}$$

Thereby, the second term of the right-hand side in (2.9) can be absorbed. Furthermore, we note the non-positivity of the integrand $\theta_\varepsilon^{d-1}\partial_t\theta_\varepsilon|\nabla c_\varepsilon|^2$ due to $\partial_t\theta_\varepsilon \le 0$ such that the corresponding integral can be omitted and finally we have

$$\left\|\mathbb{D}(\theta_\varepsilon)^{\frac{1}{2}}\nabla c_\varepsilon(t)\right\|_2^2 + \int_0^t\left\|\theta_\varepsilon^{\frac{1}{2}}\partial_t c_\varepsilon\right\|_2^2 \le D_0 + \int_0^t\left\|\theta_\varepsilon^{\sigma-\frac{1}{2}}c_\varepsilon\right\|_2^2 \le D_0 + |\Omega|T\rho^2$$

with an appropriate uniform upper bound $D_0 \ge \left\|\mathbb{D}(\theta_{0,\varepsilon})^{\frac{1}{2}}\nabla c_0\right\|_2^2$. □

Remark 1. Since the weighted time derivative $\theta_\varepsilon^{\frac{1}{2}}\partial_t c_\varepsilon$ belongs to $L^2(0,T;L^2(\Omega))$, testing $(2.4)_1$ with $\theta_\varepsilon^{-1}\nabla\cdot(\mathbb{D}(\theta_\varepsilon)\nabla c_\varepsilon)$ yields

$$\tfrac{1}{2}\int_0^t\left\|\theta_\varepsilon^{-\frac{1}{2}}\nabla\cdot(\mathbb{D}(\theta_\varepsilon)\nabla c_\varepsilon)\right\|_2^2 \le C\int_0^t\left(\left\|\theta_\varepsilon^{-\frac{1}{2}}f_\varepsilon(c_\varepsilon,\theta_\varepsilon)\right\|_2^2 + \left\|\theta_\varepsilon^{\frac{1}{2}}\partial_t c_\varepsilon\right\|_2^2\right) < \infty , \tag{2.10}$$

i.e., also $\theta_\varepsilon^{-\frac{1}{2}}\nabla\cdot(\mathbb{D}(\theta_\varepsilon)\nabla c_\varepsilon)$ lies in $L^2(0,T;L^2(\Omega))$ and hence c_ε solves the transport equation (1.1a) in a strong sense.

Henceforth, for the sake of simplicity, we assume $d = 1$. If the gradient of the initial data $\theta_{0,\varepsilon}$ is provided with more θ-weighted integrability, i.e., $\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon} \in L^4(\Omega)$ for $n = 2$ and $\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon} \in L^6(\Omega)$ for $n = 3$ (abbreviated as $\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon} \in L^{2n}(\Omega)$ below), the solution $(c_\varepsilon,\theta_\varepsilon)$ of the previous Lemma satisfies further regularity properties in certain θ-weighted norms, especially the condition (2.3a):

Lemma 2 [Regularity]. *Assume that the conditions of Lemma 1 are satisfied, such that $\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon} \in L^{2n}(\Omega)$. Then, the solution $(c_\varepsilon,\theta_\varepsilon)$ to* (2.4) *even satisfies* (2.3a) *and the regularity properties*

$$\theta_\varepsilon c_\varepsilon \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \quad \text{and} \quad \theta_\varepsilon \in L^\infty(0,T;H^2(\Omega)) .$$

Moreover, the corresponding norms are uniformly bounded with respect to $\varepsilon > 0$.

Proof. See Section 4. □

As a consequence, we conclude that every strong solution satisfies (2.3) or at least (2.3a) for $n = 2$ or $n = 3$, respectively, cf. remark below the proof.

The solutions $(c_\varepsilon,\theta_\varepsilon)_{\varepsilon>0}$ are uniformly bounded in the introduced θ-weighted norms. Thus, a compactness argument yields limit functions (c,θ) solving the original degenerating equations (1.1) in a strong sense. In particular, these limit functions inherit sufficient regularity and hence satisfy condition (2.3) or (2.3a), respectively.

Theorem 3 [Existence]. *Let* $\sigma \geq 2$, $c_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$, $\|c_0\|_\infty \leq \rho$ *and* $\theta_0 \in H^2(\Omega)$ *with* $\theta_0(x) \in [0,1)$ *for a.e.* $x \in \Omega$ *and* $\theta_0^{-1}\nabla\theta_0 \in L^{2n}(\Omega)$*. Then, there exists a bounded solution* $(c,\theta) \in \mathcal{X}_\theta \times \mathcal{Y}$ *to* (2.2) *satisfying* (2.3a) *and*

$$\theta c, \theta \in \ L^2(0,T;H^2(\Omega) \cap \ H^1(0,T;L^2(\Omega)) \ .$$

Proof. We choose for every sufficiently small $\varepsilon > 0$ the initial data $\theta_{0,\varepsilon}$ in such a way that the conditions of Lemma 2 are satisfied. Moreover, we assume that $(\theta_{0,\varepsilon})_{\varepsilon>0}$ approximates the initial data θ_0 with respect to the H^2-norm and the corresponding norms $\left\|\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon}\right\|_{2n}$ are uniformly bounded. To prove the convergence of a subsequence of $(c_\varepsilon,\theta_\varepsilon)_{\varepsilon>0}$ to a solution $(c,\theta) \in \mathcal{X}_\theta \times \mathcal{Y}$ of (2.2) we apply the previous lemma and the θ-uniform estimates (2.6) and (4.3), see Section 4. In fact $(\theta_\varepsilon c_\varepsilon)_{\varepsilon>0}$ and $(\theta_\varepsilon)_{\varepsilon>0}$ are bounded in the Banach space

$$L^2(0,T;H^2(\Omega)) \ \cap \ L^\infty(0,T;H^1(\Omega)) \ \cap \ L^\infty(\Omega_T) \ . \tag{2.11}$$

Therefore, there are subsequences $(\theta_k c_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ converging weakly* to some limits ζ and θ, respectively. These limits are bounded

$$\|\zeta\|_{L^\infty(\Omega_T)} \leq \liminf_{k\to\infty} \|\theta_k c_k\|_\infty \leq \|c_0\|_\infty \, , \ \ \|\theta\|_{L^\infty(\Omega_T)} \leq \liminf_{k\to\infty} \|\theta_k\|_\infty \leq 1.$$

With Lemma 1 we infer that also the time derivatives $\partial_t(\theta_\varepsilon c_\varepsilon)$ and $\partial_t\theta_\varepsilon$ are uniformly bounded in $L^2(0,T;L^2(\Omega))$. Then the Lemma of Aubin-Lions implies strong convergence of $(\theta_k c_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ in $C(0,T;H^1(\Omega))$ to ζ and θ, respectively. In particular, due to the boundedness of $\sup_{t\in(0,T)} \|c_\varepsilon(t)\|_\infty \leq \rho$ there is a subsequence of $(c_k)_{k\in\mathbb{N}}$ weakly* converging to a limit $c \in L^\infty(\Omega_T)$. That means with respect to $L^2(0,T;L^2(\Omega))$ the sequence of products $(\theta_k c_k)_{k\in\mathbb{N}}$ converges weakly to the product θc and hence $\theta c = \zeta$ belongs to (2.11).

Furthermore, there exists a subsequence $(\theta_k^{\sigma-1})_{k\in\mathbb{N}}$ weakly converging to $\theta^{\sigma-1}$. The limit θ actually satisfies (2.2b) since

$$\int_0^T (\partial_t(\theta_k - \theta), \varphi)_2 \ \overset{k\to\infty}{\longrightarrow} \ 0, \quad \int_0^T (f_{\varepsilon(k)}(c_k,\theta_k) - f(c,\theta), \varphi)_2$$
$$\leq \int_0^T \int_\Omega (\theta_k^{\sigma-1} - \theta^{\sigma-1})\theta c\varphi + \left\|\theta_k^{\sigma-1}\right\|_\infty \|\theta_k c_k - \theta c\|_2 \, \|\varphi\|_2 \ \overset{k\to\infty}{\longrightarrow} \ 0.$$

Similarly the convergence of the terms in (2.2a) associated with the right-hand side also holds true.

Due to possible degeneration of θ, it is not to be expected that $(\nabla c_k)_{k\in\mathbb{N}}$ and $(\partial_t c_k)_{k\in\mathbb{N}}$ converge in $L^2(\Omega_T)$ to ∇c and $\partial_t c$, respectively. Nevertheless, in what follows we verify that at least $(\theta_k\nabla c_k)_{k\in\mathbb{N}}$ and $(\theta_k\partial_t c_k)_{k\in\mathbb{N}}$ converge weakly:

The time derivatives $\theta_k\partial_t c_k$ belong to $L^2(0,T;L^2(\Omega))$, see Lemma 1, and are uniformly bounded with respect to this function space. Thus a subsequence of $(\theta_k\partial_t c_k)_{k\in\mathbb{N}}$ converges weakly and, moreover, there holds

$$\theta_k\partial_t c_k = \partial_t(\theta_k c_k) - (\partial_t\theta_k)c_k = \partial_t(\theta_k c_k) + \tfrac{1}{\rho}\theta_k^\sigma c_k^2 \rightharpoonup \partial_t(\theta c) + \tfrac{1}{\rho}\theta^\sigma c^2 \ . \tag{2.12}$$

Furthermore, the spatio-temporal distributional equation

$$\begin{aligned}\langle \theta\partial_t c, \eta\rangle_{H^{-1}(\Omega_T),H^1(\Omega_T)} &= -\, (c, \partial_t(\theta\eta))_{L^2(\Omega_T)} = -(c, (\partial_t\theta)\eta)_{L^2(\Omega_T)} \\ -(c, \theta\partial_t\eta)_{L^2(\Omega_T)} &= (\tfrac{1}{\rho}\theta^\sigma c^2, \eta)_{L^2(\Omega_T)} + (\partial_t(\theta c), \eta)_{L^2(\Omega_T)} \qquad (2.13)\end{aligned}$$

with respect to the time derivative of $c \in L^2(0,T;L^2(\Omega)) = L^2(\Omega_T)$ holds for all $\eta \in C_0^\infty(\Omega_T)$ and thus $\theta\partial_t c \in H^{-1}(\Omega_T)$ coincides with $\partial_t(\theta c) + \frac{1}{\rho}\theta^\sigma c^2 \in L^2(0,T;L^2(\Omega))$. Therefore, we conclude the weak convergence of $(\theta_k\partial_t c_k)_{k\in\mathbb{N}}$ to $\theta\partial_t c \in L^2(0,T;L^2(\Omega))$ and with (2.12) the identity $\theta\partial_t c = \partial_t(\theta c) - (\partial_t\theta)c$ in the L^2-sense.

Let us note the strong convergence of $(\theta_k)_{k\in\mathbb{N}}$ in $C(0,T;H^1(\Omega))$. Together with the weakly* convergence of $(c_k)_{k\in\mathbb{N}}$ to θ in $L^\infty(\Omega_T)$ we conclude that the sequence of products $(\nabla\theta_k \cdot c_k)_{k\in\mathbb{N}}$ converge weakly in $L^2(0,T;L^2(\Omega))$. Similarly to the time derivative we may argue with Lemma 1 that $(\theta_k\nabla c_k)_{k\in\mathbb{N}}$ converges weakly to $\theta\nabla c \in L^2(0,T;L^2(\Omega))$, even though, ∇c is not necessarily a L^2-function. Then this implies the identity $\theta\nabla c = \nabla(\theta c) - \nabla\theta\cdot c$.

Considering the terms associated with the time and spatial derivative we actually have

$$\int_0^T (\theta_k\partial_t c_k - \theta\partial_t c, \varphi)_2 \overset{k\to\infty}{\longrightarrow} 0 \ \text{ and } \int_0^T (\mathbb{D}(\theta_k)\nabla c_k - \mathbb{D}(\theta)\nabla c, \nabla\varphi)_2 \overset{k\to\infty}{\longrightarrow} 0$$

for all $\varphi \in L^2(0,T;H_0^1(\Omega))$ and therewith the existence of a solution (c,θ) to the degenerate Equations (2.2).

Since the limits $\theta c, \theta$ belong to (2.11) as well, the solution satisfies in particular (2.3a): Testing (2.2b) with

$$-\nabla\cdot(\theta^{-2n}|\nabla\theta|^{2n-2}\nabla\theta) = 2n\theta^{-2n-1}|\nabla\theta|^{2n} - (2n-1)\theta^{-2n}|\nabla\theta|^{2n-2}\Delta\theta\ ,$$

$n = 2, 3$, would formally lead to $\theta^{-1}\nabla\theta \in L^\infty(L^{2n})$, cf. end of the proof of Lemma 2 in Section 4 below. To be more rigorous, it is necessary to introduce an appropriate cut-off function $\theta_\delta := \max\{\theta, \delta\}$, $\delta > 0$. With this we actually test (2.2b) by $-\nabla\cdot(\theta_\delta^{-2n}|\nabla\theta|^{2n-2}\nabla\theta)$, which is an appropriate approximation of the above function and leads with Gronwall's Lemma (see [11, Prop. 3.4]) to the following uniformly bounded estimate with respect to $\delta > 0$:

$$\begin{aligned}\sup_{t\in(0,T)} \left\|\theta_\delta^{-1}\nabla\theta(t)\right\|_{2n} &\le \left[\left\|\theta_0^{-1}\nabla\theta_0\right\|_{2n} + \int_0^t \left\|\theta^{\sigma-2}\right\|_\infty \|\theta\nabla c\|_{2n}\right] \\ &\times \exp\left(\int_0^t \left\|\theta^{-1}\partial_t\theta\right\|_\infty + \sigma\int_0^t \left\|\theta^{\sigma-1}c\right\|_\infty\right)\ .\end{aligned}$$

Moreover, due to (2.11) we have $\nabla(\theta c), \nabla\theta \in L^2(0,T;H^1(\Omega))$ and hence

$$\theta\nabla c = \nabla(\theta c) - \nabla\theta\cdot c \in L^2(0,T;L^q(\Omega)) \qquad (2.14)$$

with $q < \infty$ for $n = 2$ and $q = 6$ for $n = 3$, respectively. Finally, taking the limit $\delta \to 0$ yields the wanted condition (2.3a). $\square$

Owing to (2.14), we obtain boundedness of $\sup_t \left\|\theta^{-1}\nabla\theta(t)\right\|_6$ in three dimensions. In fact, Theorem 2 requires slightly more regularity (2.3b) to ensure uniqueness in three dimensions. However, similar to the solutions of (2.4), the degenerating solutions (c,θ) of Theorem 3 even fulfill (2.3) for $n = 2$ since (2.14) even holds for any $q < \infty$. In this case, uniqueness of (c,θ) follows directly from Theorem 2.

3 Proof of Theorem 2

Without loss of generality we focus on the three-dimensional case, i.e. $n = 3$. Moreover, we set the density $\rho = 1$. Let $(c_1, \theta_1), (c_2, \theta_2) \in \mathcal{X}_\theta \times \mathcal{Y}$ be two weak solutions of (2.2) to the initial data (c_0, θ_0) additionally satisfying (2.3a). The solution (c_2, θ_2) is furthermore assumed to fulfill (2.3b). We set

$$(\bar{c}, \bar{\theta}) := (c_1 - c_2, \theta_1 - \theta_2) \ .$$

Boundedness of the porosities' ratio: We start with testing the ODE (2.2b) by $\theta_{2,\delta}^{-a}\bar{\theta}^{a-1}$, for $\theta_{2,\delta} := \max\{\theta_2, \delta\}$, $\delta > 0$ and some $a \in 2\mathbb{N}_0$, which yields due to

$$\int_0^t \int_\Omega \partial_t\bar{\theta} \cdot \theta_{2,\delta}^{-a}\bar{\theta}^{a-1} = \tfrac{1}{a}\left\|\theta_{2,\delta}^{-1}\bar{\theta}(t)\right\|_a^a - \tfrac{1}{a}\int_0^t \int_\Omega (\partial_t\theta_{2,\delta}^{-a})\bar{\theta}^a$$

and $f(c_i, \theta_i) = \theta_i^\sigma c_i$ for $i = 1, 2$, the inequality

$$\tfrac{1}{a}\left\|\theta_{2,\delta}^{-1}\bar{\theta}(t)\right\|_a^a \le \int_0^t \int_\Omega \left|\theta_{2,\delta}^{-a-1}(\partial_t\theta_{2,\delta})\bar{\theta}^a\right| + \int_0^t \int_\Omega \left| \left[(\theta_1^\sigma - \theta_2^\sigma)c_1 + \theta_2^\sigma\bar{c}\right]\theta_{2,\delta}^{-a}\bar{\theta}^{a-1}\right|$$
$$\le \int_0^t \left\|\theta_{2,\delta}^{-1}\partial_t\theta_{2,\delta}\right\|_\infty \left\|\theta_{2,\delta}^{-1}\bar{\theta}\right\|_a^a + \sigma\int_0^t \|c_1\|_\infty \left\|\theta_{2,\delta}^{-1}\bar{\theta}\right\|_a^a + \int_0^t \left\|\theta_2^{\sigma-1}\right\|_\infty \|\bar{c}\|_\infty \left\|\theta_{2,\delta}^{-1}\bar{\theta}\right\|_a^{a-1}.$$

Here we have applied the following estimate (3.1) of $\theta_1^\sigma - \theta_2^\sigma$: since $\theta_1, \theta_2 \in [0,1)$, there holds with $\frac{d}{d\xi}(\xi^\sigma) \le \sigma \max_{\kappa\in[0,1]}(\kappa\theta_1 + (1-\kappa)\theta_2)^{\sigma-1} =: \sigma\theta_{\max}^{\sigma-1}$ for ξ in between θ_1, θ_2 and hence $\frac{|\theta_1^\sigma - \theta_2^\sigma|}{|\theta_1 - \theta_2|} \le \sigma\theta_{\max}^{\sigma-1}$ the inequality

$$|\theta_1^\sigma - \theta_2^\sigma| \le \sigma\theta_{\max}^{\sigma-1}\bar{\theta} \ . \tag{3.1}$$

We actually obtain with $\left\|\theta_{2,\delta}^{-1}\partial_t\theta_{2,\delta}\right\|_\infty \le \left\|\theta_2^{-1}\partial_t\theta_2\right\|_\infty$ and Gronwall's Lemma

$$\sup_{t\in(0,T)} \left\|\theta_{2,\delta}^{-1}\bar{\theta}(t)\right\|_a \le \left[\int_0^T \left\|\theta_2^{\sigma-1}\right\|_\infty \|\bar{c}\|_\infty\right] \exp\left(\tfrac{1}{a}\int_0^T \left(\left\|\theta_2^{-1}\partial_t\theta_2\right\|_\infty + \sigma\|c_1\|_\infty\right)\right),$$

i.e., $\sup_t \left\|\theta_{2,\delta}^{-1}\bar{\theta}(t)\right\|_a < \infty$. Taking the limit $a \to \infty$ verifies the boundedness of $\sup_{t\in(0,T)} \left\|\theta_{2,\delta}^{-1}\bar{\theta}(t)\right\|_\infty$. Since the upper bound is independent of δ, there exists a weakly* converging subsequence of $(\theta_{2,\delta}^{-1}\bar{\theta})_{\delta>0}$ (again indexed by $\delta > 0$) with a limit ς in $L^\infty(\Omega_T)$. Moreover, the strong convergence $\theta_{2,\delta} \overset{\delta\to 0}{\longrightarrow} \theta_2$ leads to

$$\bar{\theta} = \theta_{2,\delta} \cdot (\theta_{2,\delta}^{-1}\bar{\theta}) \ \overset{\delta\to 0}{\longrightarrow} \ \theta_2 \cdot \varsigma \ ,$$

i.e., $\theta_2^{-1}\bar\theta = \varsigma \in L^\infty(\Omega_T)$. We note that the measure of $\tilde\Omega(t) := \{x \in \Omega \,|\, \theta_1(x,t) \neq 0 = \theta_2(x,t)\}$ is zero for $t \in (0,T)$. Otherwise we would have

$$\sup_{t\in(0,T)} \left\|\theta_{2,\delta}^{-1}\bar\theta(t)\right\|_\infty \geq \sup_{t\in(0,T)} \tfrac{1}{\delta}\left\|\theta_1(t)\right\|_{L^\infty(\tilde\Omega)} ,$$

where the right-hand side is unbounded for $\delta \to 0$. Finally, this leads for $\delta \to 0$ to the boundedness of

$$\sup_{t\in(0,T)} \left\|\theta_2^{-1}\bar\theta(t)\right\|_\infty = \sup_{t\in(0,T)} \left\|\tfrac{\theta_1}{\theta_2}(t) - 1\right\|_\infty , \tag{3.2}$$

i.e., θ_1/θ_2 is uniformly bounded. Changing the roles of θ_1 and θ_2 also shows the uniform boundedness of the reciprocal θ_2/θ_1. In particular, both values vanish similarly, and hence the corresponding equations degenerate in the same way. Therefore, the weighted spaces $V_{\theta_i}(\Omega)$ (and also the dual $(V_{\theta_i}(\Omega))^*$), $i = 1,2$, coincide since the corresponding norms are equivalent.

Moreover, we have in particular $\|\theta_i^{-1}\partial_t\theta_i\|_\infty < \infty$, $i = 1,2$, due to $\sigma \geq 2$ and $\theta_i^{-1}\partial_t\theta_i = \theta_i^{\sigma-1}c_i \in L^\infty(\Omega_T)$.

Inequalities of $\bar c$ in appropriate norms: Testing each single term of the transport equation (2.2a) associated to c_1 and c_2, respectively, with $\theta_1\bar c$ yields

$$\begin{aligned}
&\int_0^t \langle\theta_1\partial_t c_1 - \theta_2\partial_t c_2, \theta_1\bar c\rangle = \tfrac12\int_\Omega |\theta_1\theta_2\bar c^2(t)|\,dx - \tfrac12\int_0^t\int_\Omega \partial_t(\theta_1\theta_2)\bar c^2 + \int_0^t\langle\bar\theta\partial_t c_1, \theta_1\bar c\rangle,\\
&\int_0^t\int_\Omega (\mathbb{D}(\theta_1)\nabla c_1 - \mathbb{D}(\theta_2)\nabla c_2)\,\nabla(\theta_1\bar c) = \int_0^t\int_\Omega \theta_1\mathbb{D}(\theta_2)|\nabla\bar c|^2 + \int_0^t\int_\Omega \mathbb{D}(\theta_2)\nabla\bar c\nabla\theta_1\cdot\bar c\\
&\quad + \int_0^t\int_\Omega \theta_1\left(\mathbb{D}(\theta_1) - \mathbb{D}(\theta_2)\right)\nabla c_1\nabla\bar c + \int_0^t\int_\Omega \left(\mathbb{D}(\theta_1) - \mathbb{D}(\theta_2)\right)\nabla c_1\nabla\theta_1\cdot\bar c,\\
&\int_0^t\int_\Omega ((c_1-1)\theta_1^\sigma c_1 - (c_2-1)\theta_2^\sigma c_2)\,\theta_1\bar c = \int_0^t\int_\Omega \left[(c_1^2 - c_2^2)\theta_2^\sigma + c_1^2(\theta_1^\sigma - \theta_2^\sigma)\right]\theta_1\bar c\\
&\quad - \int_0^t\int_\Omega \left[(\theta_1^\sigma - \theta_2^\sigma)c_1 + \theta_2^\sigma\bar c\right]\theta_1\bar c.
\end{aligned} \tag{3.3}$$

Here, the product rule $\nabla(\theta_1\bar c) = (\nabla\theta_1)\bar c + \theta_1\nabla\bar c$ and similar equations are applied in the sense of distributions, cf. (2.13).

Thereby, the term $\int_0^t\langle\bar\theta\partial_t c_1, \theta_1\bar c\rangle$ will be replaced by an adequate term, cf. (3.4) below, since the direct estimation

$$\int_0^t\langle\bar\theta\partial_t c_1, \theta_1\bar c\rangle \leq \int_0^t\int_\Omega \|\theta_1\partial_t c_1\|_{V_{\theta_1}^*}\left\|\bar\theta\bar c\right\|_{V_{\theta_1}}$$

needs the control of the inconvenient norm: $\left\|\bar\theta\bar c\right\|_{V_{\theta_1}}^2 = \left\|\bar\theta\bar c\right\|_2^2 + \left\|\mathbb{D}(\theta_1)^{\frac12}\nabla(\bar\theta\bar c)\right\|_2^2$. Therefore we replace the problematic term $\int_0^t\langle\bar\theta\partial_t c_1, \theta_1\bar c\rangle$ with (2.2a) by

$$\int_0^t\langle\theta_1\partial_t c_1, \bar\theta\bar c\rangle = -\int_0^t\int_\Omega \mathbb{D}(\theta_1)\nabla c_1\nabla(\bar\theta\bar c) + \int_0^t\int_\Omega (c_1-1)\theta_1^\sigma c_1\bar\theta\bar c \tag{3.4}$$

such that the appearing derivatives can be arranged more uniformly. This leads to useful estimates which can be managed with (2.3). Combining (3.3) with (3.4) yields

$$\begin{aligned}
&\tfrac{1}{2}\left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}(t)\right\|_2^2+\int_0^t\left\|\theta_1^{\frac{1}{2}}\mathbb{D}(\theta_2)^{\frac{1}{2}}\nabla\bar{c}\right\|_2^2=\tfrac{1}{2}\int_0^t\int_\Omega\partial_t(\theta_1\theta_2)\bar{c}^2+\int_0^t\int_\Omega\mathbb{D}(\theta_1)\nabla c_1\nabla(\bar{\theta}\bar{c})\\
&\quad-\int_0^t\int_\Omega(c_1-1)\theta_1^\sigma c_1\bar{\theta}\bar{c}-\int_0^t\int_\Omega\theta_1\left(\mathbb{D}(\theta_1)-\mathbb{D}(\theta_2)\right)\nabla c_1\nabla\bar{c}\\
&\quad-\int_0^t\int_\Omega\mathbb{D}(\theta_2)\nabla\bar{c}\nabla\theta_1\cdot\bar{c}-\int_0^t\int_\Omega\left(\mathbb{D}(\theta_1)-\mathbb{D}(\theta_2)\right)\nabla c_1\nabla\theta_1\cdot\bar{c}\\
&\quad+\int_0^t\int_\Omega\left[(c_1^2-c_2^2)\theta_2^\sigma+c_1^2(\theta_1^\sigma-\theta_2^\sigma)\right]\theta_1\bar{c}-\int_0^t\int_\Omega\left[(\theta_1^\sigma-\theta_2^\sigma)c_1+\theta_2^\sigma\bar{c}\right]\theta_1\bar{c}\,.
\end{aligned}$$

The very last term on the right-hand side $-\int_0^t\int_\Omega\theta_2^\sigma\theta_1\bar{c}^2$ is negative and hence can be neglected in the following. Moreover, we apply (3.1) and obtain the inequality

$$\begin{aligned}
&\tfrac{1}{2}\left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}(t)\right\|_2^2+\int_0^t\left\|\theta_1^{\frac{1}{2}}\mathbb{D}(\theta_2)^{\frac{1}{2}}\nabla\bar{c}\right\|_2^2\le\tfrac{1}{2}\int_0^t\Big(\left\|\theta_2^{\frac{1}{2}}/\theta_1^{\frac{1}{2}}\right\|_\infty\|\partial_t\theta_1\|_\infty+\left\|\theta_1^{\frac{1}{2}}/\theta_2^{\frac{1}{2}}\right\|_\infty\\
&\quad\times\|\partial_t\theta_2\|_\infty\Big)\left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}\right\|_2^2+\int_0^t\|\theta_1\nabla c_1\|_6\Big(\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2\left\|\theta_1^{\frac{d+1}{2}}\bar{c}\right\|_3+\left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_3\\
&\quad\times\left\|\theta_1^{\frac{d+1}{2}}\nabla\bar{c}\right\|_2\Big)+\int_0^t\|(c_1-1)\theta_1^\sigma c_1\|_\infty\left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}\right\|_2\left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}\right\|_2\\
&\quad+d\int_0^t\|\theta_1\nabla c_1\|_6\left\|\theta_{\max}^{d-1}\theta_1^{-d+1}\right\|_\infty\left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_3\left\|\theta_1^{\frac{d+1}{2}}\nabla\bar{c}\right\|_2\\
&\quad\overset{(\star)}{+}\int_0^t\left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{d}{2}}\nabla\bar{c}\right\|_2\left\|\theta_1^{-1}\nabla\theta_1\right\|_6\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{d}{2}}\bar{c}\|_3\\
&\quad+d\int_0^t\left\|\theta_{\max}^{d-1}\theta_1^{-d+1}\right\|_\infty\left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_3\|\theta_1\nabla c_1\|_6\left\|\theta_1^{-1}\nabla\theta_1\right\|_6\left\|\theta_1^{\frac{d+1}{2}}\bar{c}\right\|_3\\
&\quad+\int_0^t(\|c_1\|_\infty+\|c_2\|_\infty)\left\|\theta_2^{\sigma-1}\right\|_\infty\left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}\right\|_2^2\\
&\quad+\sigma\int_0^t\left\|\theta_{\max}^{\sigma-1}\right\|_\infty\|\theta_1c_1\|_\infty(\|c_1\|_\infty+1)\left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}\right\|_2\left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}\right\|_2\,. \qquad (3.5)
\end{aligned}$$

Let us note that $\frac{d-3}{2}\ge -1$ since $d\ge 1$.

Inequalities of appropriate norms of $\bar{\theta}$: The above estimate (3.5) shows the need of controlled L^2-norms for $\theta_1^{-1}\theta_2^{-1}\bar{\theta}$ and $\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}$. Therefore, we formally test the ODE (2.2b) with $\theta_1^{-1}\theta_2^{-1}\bar{\theta}$ leading to

$$\int_0^t\int_\Omega\partial_t\bar{\theta}\cdot\theta_1^{-1}\theta_2^{-1}\bar{\theta}=\tfrac{1}{2}\left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}(t)\right\|_2^2-\tfrac{1}{2}\int_0^t\int_\Omega\partial_t(\theta_1^{-1}\theta_2^{-1})\bar{\theta}^2\,,$$

which implies with (3.1) the inequality

$$\tfrac{1}{2}\left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}(t)\right\|_2^2\le\tfrac{1}{2}\int_0^t\int_\Omega|\theta_1^{-2}(\partial_t\theta_1)\theta_2^{-1}\bar{\theta}^2|+\tfrac{1}{2}\int_0^t\int_\Omega|\theta_1^{-1}\theta_2^{-2}(\partial_t\theta_2)\bar{\theta}^2|$$

$$+\int_0^t\int_\Omega |\left[(\theta_1^\sigma-\theta_2^\sigma)c_1+\theta_2^\sigma\bar{c}\right]\theta_1^{-1}\theta_2^{-1}\bar{\theta}|$$
$$\leq \int_0^t \left(\left\|\theta_1^{-1}\partial_t\theta_1\right\|_\infty+\left\|\theta_2^{-1}\partial_t\theta_2\right\|_\infty\right)\left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}\right\|_2^2+\sigma\int_0^t \left\|\theta_{\max}^{\sigma-1}\right\|_\infty \|c_1\|_\infty$$
$$\times\left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}\right\|_2^2+\int_0^t \left\|\theta_1^{-1}\theta_2^{\sigma-1}\right\|_\infty\left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}\right\|_2\left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}\right\|_2 . \tag{3.6}$$

In order to obtain this estimate rigorously, one has to introduce cut-off functions and prove uniform boundedness similar to (3.2). Moreover, due to the result (3.2), a further derivation of such an inequality as (3.6) would have been unnecessary, but in (3.6) we work with symmetric θ_1- and θ_2-weights such that the conclusion $\theta_1=\theta_2$ at the end of the proof is more obvious.

Beside this also an appropriate estimate for $\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}$ is needed in (3.5). Thus let us formally test (2.2b) with $-\nabla\cdot(\theta_1^{d-3}\nabla\bar{\theta})$:

$$\tfrac{1}{2}\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}(t)\right\|_2^2=\tfrac{d-3}{2}\int_0^t\int_\Omega \theta_1^{d-4}\partial_t\theta_1|\nabla\bar{\theta}|^2-\int_0^t\int_\Omega \nabla\cdot\left(f(c_1,\theta_1)-f(c_2,\theta_2)\right)$$
$$\times\,\theta_1^{d-3}\nabla\bar{\theta}=\tfrac{d-3}{2}\int_0^t\int_\Omega \theta_1^{d-4}\partial_t\theta_1|\nabla\theta|^2+\int_0^t\int_\Omega\left[(\theta_1^\sigma-\theta_2^\sigma)\nabla c_1+\theta_2^\sigma\nabla\bar{c}\right]\theta_1^{d-3}\nabla\bar{\theta}$$
$$+\sigma\int_0^t\left[(\theta_1^{\sigma-1}-\theta_2^{\sigma-1})\nabla\theta_1c_1+\theta_2^{\sigma-1}\nabla\bar{\theta}c_1+\theta_2^{\sigma-1}\nabla\theta_2\bar{c}\right]\theta_1^{d-3}\nabla\bar{\theta} .$$

The right-hand side can be estimated with (3.1) (which can be used similarly for the exponent $\sigma-1\geq 1$) by

$$\tfrac{1}{2}\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}(t)\right\|_2^2\leq\tfrac{|d-3|}{2}\int_0^t\left\|\theta_1^{-1}\partial_t\theta_1\right\|_\infty\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2^2$$
$$+\sigma(\sigma-1)\int_0^t\left\|\theta_{\max}^{\sigma-2}\right\|_\infty\left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_3\left\|\theta_1^{-1}\nabla\theta_1\right\|_6\|\theta_1c_1\|_\infty\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2$$
$$+\sigma\int_0^t\left\|\theta_2^{\sigma-1}c_1\right\|_\infty\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2^2+\sigma\int_0^t\left\|\theta_2^{-1}\nabla\theta_2\right\|_6\left\|\theta_2^\sigma\theta_1^{-2}\right\|_\infty$$
$$\times\left\|\theta_1^{\frac{d+1}{2}}\bar{c}\right\|_3\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2+\sigma\int_0^t\left\|\theta_{\max}^{\sigma-1}\theta_1^{-1}\right\|_\infty\left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_3\|\theta_1\nabla c_1\|_6\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2$$
$$+\int_0^t\left\|\theta_2^{\sigma-\frac{d}{2}}\theta_1^{\frac{d}{2}-2}\right\|_\infty\left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{d}{2}}\nabla\bar{c}\right\|_2\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2 . \tag{3.7}$$

In order to control the norms $\left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_3$ and $\left\|\theta_1^{\frac{d+1}{2}}\bar{c}\right\|_3$ appearing in (3.5) and (3.7), we apply the Sobolev embedding $W^{1,\frac{3}{2}}(\Omega)\hookrightarrow L^3(\Omega)$ and Young's inequality ($\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$) to obtain

$$\left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_3^{\frac{3}{2}}\lesssim\left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_{W^{1,\frac{3}{2}}}^{\frac{3}{2}}=\left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_{\frac{3}{2}}^{\frac{3}{2}}+\left\|\nabla(\theta_1^{\frac{d-3}{2}}\bar{\theta})\right\|_{\frac{3}{2}}^{\frac{3}{2}} ,$$
$$\text{where}\quad\left\|\nabla(\theta_1^{\frac{d-3}{2}}\bar{\theta})\right\|_{\frac{3}{2}}\leq\tfrac{|d-3|}{2}\left\|\theta_1^{-1}\nabla\theta_1\right\|_6\left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_2+\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_{\frac{3}{2}}$$

as well as

$$\left\|\theta_1^{\frac{d+1}{2}}\bar{c}\right\|_3^{\frac{3}{2}} \lesssim \left\|\theta_1^{\frac{d+1}{2}}\bar{c}\right\|_{W^{1,\frac{3}{2}}}^{\frac{3}{2}} = \left\|\theta_1^{\frac{d+1}{2}}\bar{c}\right\|_{\frac{3}{2}}^{\frac{3}{2}} + \left\|\nabla(\theta_1^{\frac{d+1}{2}}\bar{c})\right\|_{\frac{3}{2}}^{\frac{3}{2}} ,$$

$$\text{where} \quad \left\|\nabla(\theta_1^{\frac{d+1}{2}}\bar{c})\right\|_{\frac{3}{2}} \leq \tfrac{|d+1|}{2}\left\|\theta_1^{-1}\nabla\theta_1\right\|_6 \left\|\theta_1^{\frac{d+1}{2}}\bar{c}\right\|_2 + \left\|\theta_1^{\frac{d+1}{2}}\nabla\bar{c}\right\|_{\frac{3}{2}} .$$

Therefore, we have with $d \geq 1$ the upper bounds of the form

$$\begin{aligned} \left\|\theta_1^{\frac{d-3}{2}}\bar{\theta}\right\|_3 &\lesssim (1 + \left\|\theta_1^{-1}\nabla\theta_1\right\|_6)\left\|\theta_1^{-1}\bar{\theta}\right\|_2 + \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2 , \\ \left\|\theta_1^{\frac{d+1}{2}}\bar{c}\right\|_3 &\lesssim (1 + \left\|\theta_1^{-1}\nabla\theta_1\right\|_6)\left\|\theta_1\bar{c}\right\|_2 + \left\|\theta_1^{\frac{d+1}{2}}\nabla\bar{c}\right\|_2 . \end{aligned} \tag{3.8}$$

Uniqueness by Gronwall's Lemma: Let us abbreviatory denote $\gamma := \|\theta_1\nabla c_1\|_6 \in L^2(0,T)$, cf. (2.3a). For the sake of readability, we suppress constants in the following, which may also depend on c_i, θ_i, $i = 1, 2$. Finally, combining (2.3a), (3.5)–(3.8) and absorbing the $\left\|\theta_1^{\frac{1}{2}}\mathbb{D}(\theta_2)^{\frac{1}{2}}\nabla\bar{c}\right\|_2$-type terms by Young's inequality results in

$$\begin{aligned}
&\tfrac{1}{2}\left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}(t)\right\|_2^2 + (1-7\delta)\int_0^t \left\|\theta_1^{\frac{1}{2}}\mathbb{D}(\theta_2)^{\frac{1}{2}}\nabla\bar{c}\right\|_2^2 + \tfrac{1}{2}\left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}(t)\right\|_2^2 + \tfrac{1}{2} \\
&\times \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}(t)\right\|_2^2 \lesssim \int_0^t \left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}\right\|_2^2 + \int_0^t \gamma\Big(\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2 \left\|\theta_1\bar{c}\right\|_2 + C(\delta)\gamma\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2^2 \\
&+ C(\delta)\gamma(\left\|\theta_1^{-1}\bar{\theta}\right\|_2 + \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2)^2\Big) + \int_0^t \left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}\right\|_2 \left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}\right\|_2 + C(\delta) \\
&\times \int_0^t \gamma^2(\left\|\theta_1^{-1}\bar{\theta}\right\|_2 + \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2)^2 \overset{(\star)}{+} \tilde{C}(\delta)\int_0^t \left\|\theta_2\bar{c}\right\|_2^2 \Big(\left\|\theta_2^{\frac{d-3}{2}}\nabla\theta_2\right\|_{6+\kappa}^{\frac{6+\kappa}{3+\kappa}} + 1\Big) \\
&+ \int_0^t \gamma\left(\left\|\theta_1^{-1}\bar{\theta}\right\|_2 + \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2\right)\Big(\left\|\theta_1\bar{c}\right\|_2 + C(\delta)\gamma\big(\left\|\theta_1^{-1}\bar{\theta}\right\|_2 + \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2\big)\Big) \\
&+ \int_0^t \left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}\right\|_2^2 + \int_0^t \left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}\right\|_2 \left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}\right\|_2 \\
&+ \int_0^t \left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}\right\|_2^2 + \int_0^t \left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}\right\|_2^2 + \int_0^t \left\|\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}\bar{c}\right\|_2 \left\|\theta_1^{-\frac{1}{2}}\theta_2^{-\frac{1}{2}}\bar{\theta}\right\|_2 \\
&+ \int_0^t \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2^2 + \int_0^t \left(\left\|\theta_1^{-1}\bar{\theta}\right\|_2 + \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2\right)\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2 \\
&+ \int_0^t \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2^2 + \int_0^t \left(\left\|\theta_1\bar{c}\right\|_2 + C(\delta)\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2\right)\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2 \\
&+ \int_0^t \gamma\left(\left\|\theta_1^{-1}\bar{\theta}\right\|_2 + \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2\right)\left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2 + C(\delta)\int_0^t \left\|\theta_1^{\frac{d-3}{2}}\nabla\bar{\theta}\right\|_2^2
\end{aligned} \tag{3.9}$$

for sufficiently small $\delta > 0$. An exception here is the term highlighted with $(\star)$ in (3.5), which is the reason for the necessity of (2.3b) since it can not be estimated by simply applying (3.8). Contrarily, this term was treated a little more subtly by using the Gagliardo-Nirenberg interpolation inequality and $(8+\kappa)/(12+2\kappa) = 1/(6+\kappa) + 1/2$

$$\left\|\theta_2^{\frac{d+1}{2}} \bar{c}\right\|_3 \lesssim \left\|\theta_2^{\frac{d+1}{2}} \bar{c}\right\|_2^{1-\frac{6+\kappa}{6+2\kappa}} \left\|\nabla(\theta_2^{\frac{d+1}{2}} \bar{c})\right\|_{\frac{12+2\kappa}{8+\kappa}}^{\frac{6+\kappa}{6+2\kappa}}$$
$$\lesssim \left\|\theta_2^{\frac{d+1}{2}} \bar{c}\right\|_2^{\frac{\kappa}{6+2\kappa}} (\left\|\theta_2^{\frac{d-3}{2}} \nabla\theta_2\right\|_{6+\kappa} \|\theta_2\bar{c}\|_2 + \left\|\theta_2^{\frac{d+1}{2}} \nabla\bar{c}\right\|_2)^{\frac{6+\kappa}{6+2\kappa}} .$$

Then Young's inequality with the conjugate exponents $\frac{12+3\kappa}{12+4\kappa} + \frac{\kappa}{12+4\kappa} = 1$ leads to

$$\int_0^t \left\|\theta_1^{\frac{1}{2}} \theta_2^{\frac{d}{2}} \nabla\bar{c}\right\|_2 \|\theta_1^{-1}\nabla\theta_1\|_6 \left\|\theta_2^{\frac{d+1}{2}} \bar{c}\right\|_3$$
$$\lesssim \int_0^t \left\|\theta_1^{\frac{1}{2}} \theta_2^{\frac{d}{2}} \nabla\bar{c}\right\|_2 \|\theta_2\bar{c}\|_2 \left\|\theta_2^{\frac{d-3}{2}} \nabla\theta_2\right\|_{6+\kappa}^{\frac{6+\kappa}{6+2\kappa}} + \int_0^t \left\|\theta_1^{\frac{1}{2}} \theta_2^{\frac{d}{2}} \nabla\bar{c}\right\|_2^{\frac{12+3\kappa}{6+2\kappa}} \|\theta_2\bar{c}\|_2^{\frac{\kappa}{6+2\kappa}}$$
$$\lesssim \tilde{C}(\delta) \int_0^t \|\theta_2\bar{c}\|_2^2 \left(\left\|\theta_2^{\frac{d-3}{2}} \nabla\theta_2\right\|_{6+\kappa}^{\frac{6+\kappa}{3+\kappa}} + 1\right) + \delta \int_0^t \left\|\theta_1^{\frac{1}{2}} \theta_2^{\frac{d}{2}} \nabla\bar{c}\right\|_2^2 .$$

Finally, with

$$\mathcal{F} := \left\|\theta_1^{\frac{1}{2}} \theta_2^{\frac{1}{2}} \bar{c}(t)\right\|_2^2 + \left\|\theta_1^{-\frac{1}{2}} \theta_2^{-\frac{1}{2}} \bar{\theta}(t)\right\|_2^2 + \left\|\theta_1^{\frac{d-3}{2}} \nabla\bar{\theta}(t)\right\|_2^2$$

we obtain by applying Gronwall's Lemma to (3.9) that $\sup_{t\in(0,T)} \mathcal{F}(t) \le 0$, i.e. $\sup_{t\in(0,T)} \mathcal{F}(t) = 0$ implying $c_1 = c_2$, $\theta_1 = \theta_2$ and hence the uniqueness of the solution. □

Remark 2. The proof for $n = 2$ is very similar to the previous one. In (3.5) and (3.7), the integrands which were managed by norms with exponents fulfilling $\frac{1}{2}+\frac{1}{3}+\frac{1}{6} = 1$, can be estimated by norms with exponents of the form $\frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 1$. That is because the embedding $W^{1,\frac{4}{3}}(\Omega) \hookrightarrow L^4(\Omega)$ is valid in the two-dimensional case, and hence estimates in the L^4-norm are sufficient. Moreover, due to $\frac{3}{4} = \frac{1}{4} + \frac{1}{2}$ we obtain inequalities similar to (3.8), including $\left\|\theta_1^{-1}\nabla\theta_1\right\|_4$. Contrarily, the embedding $W^{1,\frac{3}{2}}(\Omega) \hookrightarrow L^3(\Omega)$ and estimates with respect to the L^6-norm were necessary for three dimensions, cf. (3.8).

4 Proof of Lemma 2

For the sake of readability we suppress the index ε of the solution to (2.4) during this section, i.e. instead of $(c_\varepsilon, \theta_\varepsilon)$ we abbreviatory write (c, θ). We start by testing the modified transport equation $(2.4)_1$ with $-\Delta(\theta c)$ and obtain

$$\int_0^t \langle\nabla(\partial_t(\theta c)), \nabla(\theta c)\rangle = \tfrac{1}{2} \left[\|\nabla(\theta c)(t)\|_2^2 - \|\nabla(\theta_{0,\varepsilon} c_0)\|_2^2\right]$$

and further with $\nabla\cdot\mathbb{D}(\theta)\nabla c = \Delta(\theta c) - \Delta\theta\cdot c - \nabla\theta\nabla c$,

$$\int_0^t \int_\Omega \nabla\cdot\mathbb{D}(\theta)\nabla c\Delta(\theta c) = \int_0^t \|\Delta(\theta c)\|_2^2 - \int_0^t \int_\Omega \Delta\theta\cdot c\Delta(\theta c) - \int_0^t \int_\Omega \nabla\theta\nabla c\Delta(\theta c)$$

and finally for the reactive term: $\int_0^t\int_\Omega f_\varepsilon(c,\theta)\Delta(\theta c)\leq\int_0^t\int_\Omega|\theta^\sigma c\Delta(\theta c)|$. Note that the applied test function is admissible due to (2.7).

Applying the Laplacian Δ to the solution $\theta{=}\theta_{0,\varepsilon}-\int_0^t f_\varepsilon(c,\theta)$ of $(2.4)_4$ yields

$$\Delta\theta=\Delta\theta_{0,\varepsilon}-\int_0^t\Delta f_\varepsilon(c,\theta)\ ,$$

where for $\theta>\varepsilon$ there holds

$$\begin{aligned}\Delta f_\varepsilon(c,\theta)&=\nabla\cdot(\sigma\theta^{\sigma-1}\nabla\theta\cdot c+\theta^\sigma\nabla c)=\sigma(\sigma-1)\theta^{\sigma-2}|\nabla\theta|^2c+\sigma\theta^{\sigma-1}\Delta\theta\cdot c\\&+\sigma\theta^{\sigma-1}\nabla\theta\nabla c+(\sigma-1)\theta^{\sigma-2}\nabla\theta(\theta\nabla c)+\theta^{\sigma-1}\nabla\cdot(\theta\nabla c)=\sigma(\sigma-1)\theta^{\sigma-2}\\&\times|\nabla\theta|^2c+\sigma\theta^{\sigma-1}\Delta\theta\cdot c+(2\sigma-1)\theta^{\sigma-1}\nabla\theta\nabla c+\theta^{\sigma-\frac12}\theta^{-\frac12}\nabla\cdot(\theta\nabla c)\ .\end{aligned}$$

Thereby $\theta^{-\frac12}\nabla\cdot(\theta\nabla c)$ in the last summand of the right-hand side belongs to $L^2(0,T;L^2(\Omega))$ due to (2.10). Summarizing leads to

$$\begin{aligned}&\tfrac12\|\nabla(\theta c)(t)\|_2^2+\int_0^t\|\Delta(\theta c)\|_2^2+\|\Delta\theta(t)\|_2^2\leq\tfrac12\|\nabla(\theta_{0,\varepsilon}c_0)\|_2^2 &(4.1)\\&+\int_0^t\int_\Omega\Delta\theta\cdot c\Delta(\theta c)+\int_0^t\int_\Omega\nabla\theta\nabla c\Delta(\theta c)+\int_0^t\int_\Omega|\theta^\sigma c\Delta(\theta c)|\\&+\|\Delta\theta_{0,\varepsilon}\|_2\|\Delta\theta(t)\|_2+\sigma(\sigma-1)\int_0^t\int_\Omega\theta^{\sigma-2}|\nabla\theta|^2c\Delta\theta+\sigma\int_0^t\int_\Omega\theta^{\sigma-1}\Delta\theta\cdot c\Delta\theta\\&+(2\sigma-1)\int_0^t\int_\Omega\theta^{\sigma-1}\nabla\theta\nabla c\Delta\theta+\int_0^t\int_\Omega\theta^{\sigma-\frac12}\theta^{-\frac12}\nabla\cdot(\theta\nabla c)\Delta\theta\ . &(4.2)\end{aligned}$$

Since the gradient of c can generally not be controlled in the L^4-norm, we replace ∇c by $\theta^{-1}(\nabla(\theta c)-\nabla\theta\cdot c)$. Then an additional weight θ^{-1} occurs, which can be absorbed by $\nabla\theta$ in the L^4-norm: We test the ODE $(2.4)_4$ with $-\nabla\cdot(\theta^{-4}|\nabla\theta|^2\nabla\theta)$ leading to

$$\int_0^t\int_\Omega(\partial_t(\nabla\theta))\theta^{-4}|\nabla\theta|^2\nabla\theta=\tfrac14\left[\left\|\theta^{-1}\nabla\theta(t)\right\|_4^4-\left\|\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon}\right\|_4^4\right]+\int_0^t\int_\Omega\theta^{-5}\partial_t\theta|\nabla\theta|^4,$$

$$\int_0^t\int_\Omega\nabla f_\varepsilon(\theta,c)\cdot\theta^{-4}|\nabla\theta|^2\nabla\theta\leq\sigma\int_0^t\int_\Omega\left|\theta^{\sigma-1}\nabla\theta\cdot c\theta^{-4}|\nabla\theta|^2\nabla\theta\right|+\int_0^t\int_\Omega\left|\theta^\sigma\nabla c\theta^{-4}|\nabla\theta|^2\nabla\theta\right|$$

such that we obtain

$$\begin{aligned}&\tfrac14\left\|\theta^{-1}\nabla\theta(t)\right\|_4^4\leq\tfrac14\left\|\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon}\right\|_4^4-\int_0^t\int_\Omega\theta^{-5}\partial_t\theta|\nabla\theta|^4\\&+\sigma\int_0^t\int_\Omega\left|\theta^{\sigma-1}\nabla\theta\cdot c\theta^{-4}|\nabla\theta|^2\nabla\theta\right|+\int_0^t\int_\Omega\left|\theta^\sigma\nabla c\theta^{-4}|\nabla\theta|^2\nabla\theta\right|\ .\end{aligned}$$

Combining this with (4.2) results in

$$\begin{aligned}&\tfrac12\|\nabla(\theta c)(t)\|_2^2+\int_0^t\|\Delta(\theta c)\|_2^2+\tfrac12\|\Delta\theta(t)\|_2^2+\tfrac14\left\|\theta^{-1}\nabla\theta(t)\right\|_4^4\leq\tfrac12\|\nabla(\theta_{0,\varepsilon}c_0)\|_2^2\\&+\int_0^t\int_\Omega\Delta\theta\cdot c\Delta(\theta c)+\int_0^t\int_\Omega\theta^{-1}\nabla\theta\big(\nabla(\theta c)-\nabla\theta\cdot c\big)\Delta(\theta c)+\int_0^t\int_\Omega|\theta^\sigma c\Delta(\theta c)|\end{aligned}$$

$$+\tfrac{1}{2}\left\|\varDelta\theta_{0,\varepsilon}\right\|_2^2+\sigma(\sigma-1)\int_0^t\int_\Omega \theta^{\sigma-2}|\nabla\theta|^2 c\varDelta\theta+\sigma\int_0^t\int_\Omega \theta^{\sigma-1}\varDelta\theta\cdot c\varDelta\theta$$
$$+(2\sigma-1)\int_0^t\int_\Omega \theta^{\sigma-1}\theta^{-1}\nabla\theta\big(\nabla(\theta c)-\nabla\theta\cdot c\big)\varDelta\theta+\int_0^t\int_\Omega \theta^{\sigma-\frac{1}{2}}\theta^{-\frac{1}{2}}\nabla\cdot(\theta\nabla c)\varDelta\theta$$
$$+\tfrac{1}{4}\left\|\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon}\right\|_4^4-\int_0^t\int_\Omega \theta^{-5}\partial_t\theta|\nabla\theta|^4+\sigma\int_0^t\int_\Omega \left|\theta^{\sigma-1}\nabla\theta\cdot c\theta^{-4}|\nabla\theta|^2\nabla\theta\right|$$
$$+\int_0^t\int_\Omega \left|\theta^\sigma\theta^{-1}\big(\nabla(\theta c)-\nabla\theta\cdot c\big)\theta^{-4}|\nabla\theta|^2\nabla\theta\right|$$

such that

$$\tfrac{1}{2}\left\|\nabla(\theta c)(t)\right\|_2^2+\int_0^t\left\|\varDelta(\theta c)\right\|_2^2+\tfrac{1}{2}\left\|\varDelta\theta(t)\right\|_2^2+\tfrac{1}{4}\left\|\theta^{-1}\nabla\theta(t)\right\|_4^4\le\tfrac{1}{2}\left\|\nabla(\theta_{0,\varepsilon}c_0)\right\|_2^2$$
$$+\int_0^t\left\|\varDelta\theta\right\|_2\left\|c\right\|_\infty\left\|\varDelta(\theta c)\right\|_2+\int_0^t\left\|\theta^{-1}\nabla\theta\right\|_4\left(\left\|\nabla(\theta c)\right\|_4+\left\|\theta^{-1}\nabla\theta\right\|_4\left\|\theta c\right\|_\infty\right)$$
$$\times\left\|\varDelta(\theta c)\right\|_2+\int_0^t\left\|\theta^\sigma c\right\|_2\left\|\varDelta(\theta c)\right\|_2+\tfrac{1}{2}\left\|\varDelta\theta_{0,\varepsilon}\right\|_2^2+\sigma(\sigma-1)\int_0^t\left\|\theta^\sigma\right\|_\infty\left\|\theta^{-1}\nabla\theta\right\|_4^2$$
$$\times\left\|c\right\|_\infty\left\|\varDelta\theta\right\|_2+\sigma\int_0^t\left\|\theta^{\sigma-1}c\right\|_\infty\left\|\varDelta\theta\right\|_2^2+(2\sigma-1)\int_0^t\Big(\left\|\theta^{\sigma-1}\right\|_\infty\left\|\nabla(\theta c)\right\|_4$$
$$+\left\|\theta^\sigma c\right\|_\infty\left\|\theta^{-1}\nabla\theta\right\|_4\Big)\left\|\theta^{-1}\nabla\theta\right\|_4\left\|\varDelta\theta\right\|_2+\int_0^t\left\|\theta^{\sigma-\frac{1}{2}}\right\|_\infty\left\|\theta^{-\frac{1}{2}}\nabla\cdot(\theta\nabla c)\right\|_2\left\|\varDelta\theta\right\|_2$$
$$+\tfrac{1}{4}\left\|\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon}\right\|_4^4+(1+\sigma)\int_0^t\left\|\theta^{\sigma-1}c\right\|_\infty\left\|\theta^{-1}\nabla\theta\right\|_4^4+\int_0^t\Big(\left\|\theta^{\sigma-2}\right\|_\infty\left\|\nabla(\theta c)\right\|_4$$
$$+\left\|\theta^{\sigma-1}c\right\|_\infty\left\|\theta^{-1}\nabla\theta\right\|_4\Big)\left\|\theta^{-1}\nabla\theta\right\|_4^3.$$

Use of the Gagliardo-Nirenberg inequality entails

$$\left\|\nabla(\theta c)\right\|_4^2\le C_{GN}^2\left\|\varDelta(\theta c)\right\|_2\left\|\theta c\right\|_\infty$$

for both dimensions $n=2,3$. We actually absorb the terms $\left\|\varDelta(\theta c)\right\|_2$ on the right-hand side with Young's inequality and obtain

$$\tfrac{1}{2}\left\|\nabla(\theta c)(t)\right\|_2^2+(1-5\delta)\int_0^t\left\|\varDelta(\theta c)\right\|_2^2+\tfrac{1}{2}\left\|\varDelta\theta(t)\right\|_2^2+\tfrac{1}{4}\left\|\theta^{-1}\nabla\theta(t)\right\|_4^4$$
$$\le\tfrac{1}{2}\left\|\nabla(\theta_{0,\varepsilon}c_0)\right\|_2^2+C(\delta)\int_0^t\left\|\varDelta\theta\right\|_2^2\left\|c\right\|_\infty^2+C(\delta)(C_{GN}^4+1)\int_0^t\left\|\theta^{-1}\nabla\theta\right\|_4^4\left\|\theta c\right\|_\infty^2$$
$$+C(\delta)\int_0^t\left\|\theta^\sigma c\right\|_2^2+\tfrac{1}{2}\left\|\varDelta\theta_{0,\varepsilon}\right\|_2^2+\sigma(\sigma-1)\int_0^t\left\|\theta^\sigma\right\|_\infty\left\|\theta^{-1}\nabla\theta\right\|_4^2\left\|c\right\|_\infty\left\|\varDelta\theta\right\|_2$$
$$+\sigma\int_0^t\left\|\theta^{\sigma-1}c\right\|_\infty\left\|\varDelta\theta\right\|_2^2+C(\delta)((2\sigma-1)C_{GN})^{\frac{4}{3}}\int_0^t\left\|\theta^{\sigma-1}\right\|_\infty^{\frac{4}{3}}\left\|\theta c\right\|_\infty^{\frac{2}{3}}\left\|\theta^{-1}\nabla\theta\right\|_4^{\frac{4}{3}}$$
$$\times\left\|\varDelta\theta\right\|_2^{\frac{4}{3}}+(2\sigma-1)\int_0^t\left\|\theta^\sigma c\right\|_\infty\left\|\theta^{-1}\nabla\theta\right\|_4^2\left\|\varDelta\theta\right\|_2+\int_0^t\left\|\theta^{\sigma-\frac{1}{2}}\right\|_\infty\left\|\theta^{-\frac{1}{2}}\nabla\cdot(\theta\nabla c)\right\|_2$$
$$\times\left\|\varDelta\theta\right\|_2+\tfrac{1}{4}\left\|\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon}\right\|_4^4+(1+\sigma)\int_0^t\left\|\theta^{\sigma-1}c\right\|_\infty\left\|\theta^{-1}\nabla\theta\right\|_4^4+C(\delta)C_{GN}^{\frac{4}{3}}$$

$$\times \int_0^t \left\|\theta^{\sigma-2}\right\|_\infty^{\frac{4}{3}} \|\theta c\|_\infty^{\frac{2}{3}} \left\|\theta^{-1}\nabla\theta\right\|_4^4 + \int_0^t \left\|\theta^{\sigma-1}c\right\|_\infty \left\|\theta^{-1}\nabla\theta\right\|_4^4 .$$

Now let

$$F(c,\theta)(t) := \tfrac{1}{2}\left\|\nabla(\theta c)(t)\right\|_2^2 + \tfrac{1}{2}\left\|\Delta\theta(t)\right\|_2^2 + \tfrac{1}{4}\left\|\theta^{-1}\nabla\theta(t)\right\|_4^4 .$$

Then we have

$$\begin{aligned}
F(c,\theta)(t) \lesssim{}& \tfrac{1}{2}\left\|\nabla(\theta_{0,\varepsilon}c_0)\right\|_2^2 + 2\int_0^t F(c,\theta)\,\|c\|_\infty^2 + 8\int_0^t F(c,\theta)\,\|\theta c\|_\infty^2 + \int_0^t \|\theta^\sigma c\|_2^2 \\
&+ \tfrac{1}{2}\left\|\Delta\theta_{0,\varepsilon}\right\|_2^2 + 2\sqrt{2}\int_0^t \|\theta^\sigma\|_\infty F(c,\theta)^{\frac{1}{2}} \|c\|_\infty F(c,\theta)^{\frac{1}{2}} + 2\int_0^t \left\|\theta^{\sigma-1}c\right\|_\infty F(c,\theta) \\
&+ 4^{\frac{1}{3}}2^{\frac{2}{3}} \int_0^t \left\|\theta^{\sigma-1}\right\|_\infty^{\frac{4}{3}} \|\theta c\|_\infty^{\frac{2}{3}} F(c,\theta)^{\frac{1}{3}} F(c,\theta)^{\frac{2}{3}} + 2\sqrt{2}\int_0^t \|\theta^\sigma c\|_\infty F(c,\theta)^{\frac{1}{2}} F(c,\theta)^{\frac{1}{2}} \\
&+ \sqrt{2}\int_0^t \left\|\theta^{\sigma-\frac{1}{2}}\right\|_\infty \left\|\theta^{-\frac{1}{2}}\nabla\cdot(\theta\nabla c)\right\|_2 F(c,\theta)^{\frac{1}{2}} + \tfrac{1}{4}\left\|\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon}\right\|_4^4 + 4\int_0^t \left\|\theta^{\sigma-1}c\right\|_\infty \\
&\times F(c,\theta) + 4\int_0^t \left\|\theta^{\sigma-2}\right\|_\infty^{\frac{4}{3}} \|\theta c\|_\infty^{\frac{2}{3}} F(c,\theta) + 4\int_0^t \left\|\theta^{\sigma-1}c\right\|_\infty F(c,\theta)
\end{aligned}$$

such that Gronwall's Lemma yields boundedness of $\sup_{t\in(0,T)}\|F(c,\theta)(t)\|$

$$\begin{aligned}
\lesssim{}& \Bigg[\left(\tfrac{1}{2}\left\|\nabla(\theta_{0,\varepsilon}c_0)\right\|_2^2 + \int_0^t \|\theta^\sigma c\|_2^2 + \tfrac{1}{2}\left\|\Delta\theta_{0,\varepsilon}\right\|_2^2 + \tfrac{1}{4}\left\|\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon}\right\|_4^4\right)^{\frac{1}{2}} \\
&+ \tfrac{\sqrt{2}}{2}\int_0^t \left\|\theta^{\sigma-\frac{1}{2}}\right\|_\infty \left\|\theta^{-\frac{1}{2}}\nabla\cdot(\theta\nabla c)\right\|_2\Bigg]^2 \exp\Big(2T\sup_t \|c(t)\|_\infty^2 + 8T\sup_t\|\theta c(t)\|_\infty^2 \\
&+ 2\sqrt{2}T\sup_t(\|\theta^\sigma(t)\|_\infty \|c(t)\|_\infty) + 2T\sup_t \left\|\theta^{\sigma-1}c(t)\right\|_\infty + 4^{\frac{1}{3}}2^{\frac{2}{3}}T \\
&\times \sup_t(\left\|\theta^{\sigma-1}(t)\right\|_\infty^{\frac{4}{3}} \|\theta c(t)\|_\infty^{\frac{2}{3}}) + 2\sqrt{2}T\sup_t\|\theta^\sigma c(t)\|_\infty + 4T\sup_t\left\|\theta^{\sigma-1}c(t)\right\|_\infty \\
&+ 4T\sup_t(\left\|\theta^{\sigma-2}(t)\right\|_\infty^{\frac{4}{3}} \|\theta c(t)\|_\infty^{\frac{2}{3}}) + 4T\sup_t(\left\|\theta^{\sigma-1}(t)\right\|_\infty \|c(t)\|_\infty)\Big) \\
\lesssim{}& \left[C_0^{\frac{1}{2}} + \tfrac{1}{\sqrt{2}}\int_0^t \left\|\theta^{-\frac{1}{2}}\nabla\cdot(\theta\nabla c)\right\|_2\right]^2 \exp(C_1 T) \;<\; \infty\, . \qquad (4.3)
\end{aligned}$$

Thereby, the constants $C_0 > 0$ and $C_1 > 0$ are uniformly bounded with respect to $\varepsilon > 0$ due to (2.6) and an appropriate choice of the approximating initial data $\theta_{0,\varepsilon}$. Moreover, (2.10) ensures uniform boundedness also for the term $\left\|\theta^{-\frac{1}{2}}\nabla\cdot(\theta\nabla c)\right\|_2$. Finally, the above estimates imply $\theta c \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$ and $\theta \in L^\infty(0,T;H^2(\Omega))$.
Moreover, we have $\nabla(\theta c), \nabla\theta \in L^2(0,T;H^1(\Omega))$ and hence

$$\theta\nabla c = \nabla(\theta c) - \nabla\theta\cdot c \in L^2(0,T;L^q(\Omega)) \qquad (4.4)$$

with $q < \infty$ for $n = 2$ and $q = 6$ for $n = 3$, respectively. Since $\theta\nabla c$ belongs at least to $L^2(0,T;L^6(\Omega))$, we test $(2.4)_4$ with $-\nabla\cdot(\theta^{-6}|\nabla\theta|^4\nabla\theta)$ and obtain

$$\tfrac{1}{6}\left\|\theta^{-1}\nabla\theta(t)\right\|_6^6 \le \tfrac{1}{6}\left\|\theta_{0,\varepsilon}^{-1}\nabla\theta_{0,\varepsilon}\right\|_6^6 + \int_0^t\int_\Omega \theta^{-7}\partial_t\theta|\nabla\theta|^6$$

$$
\begin{aligned}
&+ \sigma \int_0^t \int_\Omega \theta^{\sigma-1} \nabla\theta \cdot c\theta^{-6} |\nabla\theta|^4 \nabla\theta + \int_0^t \int_\Omega \theta^{\sigma} \nabla c \theta^{-6} |\nabla\theta|^4 \nabla\theta \\
&\leq \tfrac{1}{6} \left\|\theta_{0,\varepsilon}^{-1} \nabla\theta_{0,\varepsilon}\right\|_6^6 + \int_0^t \left\|\theta^{-1}\partial_t\theta\right\|_\infty \left\|\theta^{-1}\nabla\theta\right\|_6^6 \\
&+ \sigma \int_0^t \left\|\theta^{\sigma-1} c\right\|_\infty \left\|\theta^{-1}\nabla\theta\right\|_6^6 + \int_0^t \left\|\theta^{\sigma-2}\right\|_\infty \left\|\theta\nabla c\right\|_6 \left\|\theta^{-1}\nabla\theta\right\|_6^5 .
\end{aligned}
$$

Again the Lemma of Gronwall yields the boundedness of $\theta^{-1}\nabla\theta$ in $L^\infty(0,T;L^6(\Omega))$, i.e. (c,θ) satisfies (2.3a). □

Remark 3. Because of (4.4) we obtain boundedness of $\sup_t \|\theta^{-1}\nabla\theta(t)\|_q$ for $q \leq 6$ in three dimensions, but in the two-dimensional case even for any $q < \infty$. Consequently, for $n = 2$ the solution (c,θ) of the above proof also fulfills the condition (2.3b).

5 Conclusions

The current study analyzed the diffusion-precipitation model (1.1), including vanishing porosity. Due to nonlinearity and degenerating θ-weights in this model, the proof of uniqueness is rather challenging. Introducing appropriate θ-weighted norms enabled us to handle the degeneracy. We assumed additional conditions (2.3) under which uniqueness of weak solutions was established. Moreover, a proof of existence, which is based on a compactness argument, entails rather regular solutions to (1.1) (almost) satisfying the uniqueness conditions (2.3). As a result, every two-dimensional strong solution is unique. In contrast, the slightly different condition (2.3b) is additionally required and hence uniqueness is still open in three dimensions.

Slightly more θ-weighted integrability (2.3b) would ensure uniqueness in three dimensions. However, since a $L^{\frac{6+\kappa}{3+\kappa}}(L^{6+\kappa})$-norm for some $\kappa > 0$ needs to be controlled, it may be useful to employ strong L^p-theory of parabolic equations. Although this is beyond the scope of this paper, it is nevertheless of interest for future work.

For simplicity, the diffusivity was proposed to be scalar-valued. In fact, this parameter is a tensor $\mathbb{D} : [0,1) \to \mathbb{R}^{(n,n)}$ in anisotropic media [4]. Such a generalization can be easily made since the effective tensor is typically bounded, symmetric, and still positive semidefinite in the case of clogging [8].

Although this article is one of the first steps towards rigorous analysis of clogging porous media, a lot of future work still needs to be done in various directions. For example, one should extend the range of σ and generalize the reaction rate. Moreover, fluid flow and advective transport should be incorporated for a more comprehensive model.

In fact, the restriction that the diffusivity $\mathbb{D}(\theta)$ degenerates only for $\theta = 0$ is generally not reasonable for an arbitrary geometric setting. Depending on the underlying geometry of the medium's microstructure the diffusivity may vanish for a positive critical porosity $\theta_{\text{clog}} > 0$, cf. [4]. However, assuming $\theta_{\text{clog}} = 0$, the present work avoided technical excesses due to possible post-clogging precipitation processes. Nevertheless, since the analytically justified restriction on the critical porosity $\theta_{\text{clog}} = 0$ is not exhaustive, analysis for $\theta_{\text{clog}} > 0$ is also important and should be done in the future.

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