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Strong Convergence to Common Fixed Points Using Ishikawa and Hybrid Methods for Mean-Demiclosed Mappings in Hilbert Spaces

Atsumasa Kondo

Department of Economics, Shiga University
Banba 1-1-1, Hikone, 522-0069 Shiga, Japan
E-mail: a-kondo@biwako.shiga-u.ac.jp

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Abstract. In this paper, we establish a strong convergence theorem that approximates a common fixed point of two nonlinear mappings by comprehensively using an Ishikawa iterative method, a hybrid method, and a mean-valued iterative method. The shrinking projection method is also developed. The nonlinear mappings are a general type that includes nonexpansive mappings and other classes of well-known mappings. The two mappings are not assumed to be continuous or commutative. The main theorems in this paper generate a variety of strong convergence theorems including a type of "three-step iterative method". An application to the variational inequality problem is also given.

Keywords: Ishikawa iteration, hybrid method, shrinking projection method, mean-valued iteration, mean-demiclosed mapping, common fixed point.

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1 Introduction

In this paper, the notation H represents a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. A set of all fixed points of a mapping $T : C \to H$ is denoted by

$$F(T) = \{u \in C : Tu = u\},$$

where C is a nonempty subset of H. The problem of approximating fixed points of nonlinear mappings has attracted many researchers, and many iteration methods have been proposed. The following iterative procedure is called

Mann's type [19]:

$$x_{n+1} = a_n x_n + (1 - a_n) T x_n, \tag{1.1}$$

for all $n \in \mathbb{N}$. In (1.1), $x_1 \in C$ is arbitrary and $a_n \in [0,1]$ is a parameter that satisfies certain conditions. It is well-known that this iteration yields weak convergence; for instance, Reich [24] demonstrated weak convergence to a fixed point under the iteration (1.1) in a Banach space setting. In 1974, Ishikawa [7] introduced a more general iteration than Mann's type (1.1):

$$\begin{aligned} z_n &= \lambda_n x_n + (1 - \lambda_n) T x_n, \\ x_{n+1} &= a_n x_n + (1 - a_n) T z_n, \end{aligned} \tag{1.2}$$

where x_1 is provided and $\lambda_n, a_n \in [0,1]$. If $\lambda_n = 1$ for all $n \in \mathbb{N}$ in (1.2), then, Ishikawa iteration coincides with Mann's iteration (1.1). Various convergence theorems basing on the Ishikawa iteration have been studied by many researchers; see [1, 11, 13].

Following mean-valued iterations by Baillon [3] as well as Shimizu and Takahashi [25], Atsushiba and Takahashi considered the following iteration in their 1998's paper [2]:

$$x_{n+1} = a_n x_n + (1 - a_n) \frac{1}{n^2} \sum_{l=0}^{n-1} \sum_{k=0}^{n-1} S^l T^k x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.3}$$

where $x_1 \in C$ is provided and $a_n \in [0,1]$. They demonstrated weak convergence to a common fixed point of nonexpansive mappings S and T that satisfy $ST = TS$. A mapping T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For successive studies of the mean-valued iterative method, see [1,6,13,14,16,17,18].

For a nonempty, closed, and convex subset D of H, we use P_D to represent a metric projection from H onto D. In 2003, Nakajo and Takahashi [21] proved a strong convergence theorem for finding a fixed point of a nonexpansive mapping:

Theorem 1 ([21]). *Let C be a nonempty, closed, and convex subset of H. Let T be a nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$. Let $a \in [0,1)$ and let $\{a_n\}$ be a sequence of real numbers such that $0 \leq a_n \leq a < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} & x_1 = x \in C \text{ given}, \\ & y_n = a_n x_n + (1 - a_n) T x_n \in C, \\ & C_n = \{h \in C : \|y_n - h\| \leq \|x_n - h\|\}, \\ & Q_n = \{h \in C : \langle x - x_n,\ x_n - h \rangle \geq 0\}, \\ & x_{n+1} = P_{C_n \cap Q_n} x \quad \forall n \in \mathbb{N}. \end{aligned} \tag{1.4}$$

Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(T)$, where $\widehat{x} = P_{F(T)} x$.

In 2008, Takahashi, Takeuchi, and Kubota [28] established an approximation method to find a fixed point of a nonexpansive mapping by employing metric projections on shrinking sets $\{C_n\}$. A simple version of their result is shown as follows:

Theorem 2 ([28]). *Let C be a nonempty, closed, and convex subset of H. Let T be a nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$. Let $a \in [0,1)$ and let $\{a_n\}$ be a sequence of real numbers such that $0 \leq a_n \leq a < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 = x \in C \text{ given}, \tag{1.5}$$

$$C_1 = C, \tag{1.6}$$

$$\begin{aligned} &y_n = a_n x_n + (1 - a_n) T x_n \in C, \\ &C_{n+1} = \{h \in C_n : \|y_n - h\| \leq \|x_n - h\|\}, \\ &x_{n+1} = P_{C_{n+1}} x \quad \forall n \in \mathbb{N}. \end{aligned}$$

Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(T)$, where $\widehat{x} = P_{F(T)} x$.

Following (1.3), Kondo and Takahashi [18] introduced the following mean-valued iterative procedure

$$x_{n+1} = a_n x_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n, \tag{1.7}$$

where $x_1 \in C$ is provided and $a_n, b_n, c_n \in [0,1]$ are the coefficients of a convex combination. In (1.7), S and $T : C \to C$ are nonlinear mappings of a more general type than nonexpansive mappings, and $ST = TS$ is not assumed. Kondo and Takahashi established weak convergence to a common fixed point of S and T. Strong convergence theorems using Halpern's iteration together with the mean-valued iteration was presented in another paper [17]. Very recently, Kondo [14] combined the mean-valued iteration (1.7) with the hybrid method (1.4) and the shrinking projection method (1.6), and he obtained strong convergence theorems; see Corollaries 1 and 2 in this paper. Furthermore, Kondo [13] used an Ishikawa iteration (1.2) with the mean iteration (1.7) and derived a variety of weak convergence results.

In this paper, we comprehensively use an Ishikawa iteration (1.2), a hybrid method (1.4), and a mean-valued iteration (1.7) and establish a strong convergence theorem that approximates a common fixed point of two nonlinear mappings. The shrinking projection method (1.6) is also developed. The mappings are of a general type that includes nonexpansive mappings as special cases. Similar to Kondo [13], which derived a variety of weak convergence results, the approach used in this paper generates many types of strong convergence results. At the outset, we prepare prerequisite information in Section 2. In Section 3, Nakajo and Takahashi's strong convergence theorem is obtained whereas in Section 4, Takahashi, Takeuchi and Kubota's type strong convergence is demonstrated. Our proofs do not require that the two mappings be continuous or commutative. In Section 5, various types of strong convergence results are derived from the main theorems presented in the previous two sections. In Section 6, an application to the variational inequality problem is demonstrated. Section 7 provides a concise conclusion. The possibility of extension from the class of nonexpansive mappings to include more general classes is discussed in Appendix A.

2 Preliminaries

In this section, preliminary information and results are briefly presented.

Strong convergence of a sequence $\{x_n\}$ in a real Hilbert space H to a point $x\,(\in H)$ is denoted by $x_n \to x$, whereas weak convergence is represented by $x_n \rightharpoonup x$. Strong convergence $x_n \to x$ is characterized by the following condition: for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to x$. A closed and convex set C in H is weakly closed, that is, $\{x_n\} \subset C$ and $x_n \rightharpoonup x$ imply $x \in C$. Maruyama et al. [20] showed that for $x, y, z \in H$ and $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$, the following holds:

$$\begin{aligned} &\|ax + by + cz\|^2 \qquad (2.1)\\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2. \end{aligned}$$

Let C be a nonempty, closed, and convex subset of H. Following convention, we use P_C to denote the *metric projection* from H onto C, that is, $\|x - P_C x\| \leq \|x - h\|$ for all $x \in H$ and $h \in C$. The metric projection is nonexpansive. The metric projection P_C from H onto C has the following properties:

$$\langle x - P_C x,\ P_C x - h\rangle \geq 0 \text{ and} \qquad (2.2)$$

$$\|x - P_C x\|^2 + \|P_C x - h\|^2 \leq \|x - h\|^2 \qquad (2.3)$$

for all $x \in H$ and $h \in C$.

A mapping $S : C \to H$ with $F(S) \neq \emptyset$ is said to be *quasi-nonexpansive* if

$$\|Sx - q\| \leq \|x - q\| \qquad (2.4)$$

for all $x \in C$ and $q \in F(S)$, where C is a nonempty subset of H. According to Itoh and Takahashi [8], the set of all fixed points of a quasi-nonexpansive mapping S is closed and convex. Consequently, for a quasi-nonexpansive mapping S, the metric projection $P_{F(S)}$ from H onto $F(S)$ can be considered.

Let S be a self-mapping defined on C such that $F(S) \neq \emptyset$, where C is a nonempty, closed, and convex subset of H. Let $\{z_n\}$ be a bounded sequence in C, and define $Z_n \equiv \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n\,(\in C)$ for each $n \in \mathbb{N}$. Kondo [14] called a mapping $S : C \to C$ *mean-demiclosed* if

$$Z_{n_i} \rightharpoonup v \Longrightarrow v \in F(S), \qquad (2.5)$$

where $\{Z_{n_i}\}$ is a subsequence of $\{Z_n\}$. The class of mappings that we address in this paper comprises quasi-nonexpansive and mean-demiclosed mappings. As we prove next, that class of mappings contains nonexpansive mappings. Therefore, the results in this paper have broad applicability. For an application to the variational inequality problem, see Section 6 in this paper.

Claim 1. *A nonexpansive mapping that has a fixed point is quasi-nonexpansive and mean-demiclosed.*

Proof. Let $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$. Let $x \in C$ and $q \in F(S)$. Using $q = Sq$ and the hypothesis that S is nonexpansive, we obtain

$$\|Sx - q\| = \|Sx - Sq\| \leq \|x - q\|.$$

This result shows that S is quasi-nonexpansive (2.4).

Next, we prove that $S : C \to C$ is mean-demiclosed, where C is nonempty, closed, and convex subset of H. Let $Z_n \equiv \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n \,(\in C)$, where $\{z_n\}$ is a bounded sequence in C, and let $\{Z_{n_i}\}$ be a subsequence of $\{Z_n\}$ such that $Z_{n_i} \rightharpoonup v$ for some $v \in H$. As $\{Z_{n_i}\}$ is a sequence in C and C is weakly closed, we have $v \in C$. Therefore, $Sv\,(\in C)$ is defined. Our goal is to prove that $Sv = v$. As S is nonexpansive,

$$\left\|S^{l+1} z_n - Sv\right\|^2 \le \left\|S^l z_n - v\right\|^2$$

for all $n \in \mathbb{N}$ and $l \in \mathbb{N}\cup\{0\}$. Then, it follows that

$$\left\|S^{l+1} z_n - Sv\right\|^2 \le \left\|S^l z_n - Sv\right\|^2 + 2\left\langle S^l z_n - Sv,\ Sv - v\right\rangle + \left\|Sv - v\right\|^2.$$

Summing these inequalities with respect to l from 0 to $n-1$ and dividing by n, we obtain

$$\frac{1}{n}\left\|S^n z_n - Sv\right\|^2 \le \frac{1}{n}\left\|z_n - Sv\right\|^2 + 2\left\langle Z_n - Sv,\ Sv - v\right\rangle + \left\|Sv - v\right\|^2.$$

As $\frac{1}{n}\left\|S^n z_n - Sv\right\|^2 \ge 0$, it follows that

$$0 \le \frac{1}{n}\left\|z_n - Sv\right\|^2 + 2\left\langle Z_n - Sv,\ Sv - v\right\rangle + \left\|Sv - v\right\|^2.$$

Note that $\{z_n\}$ is bounded and that $Z_{n_i} \rightharpoonup v$ is assumed. Replacing n with n_i and taking the limit as $i \to \infty$, we obtain

$$0 \le 2\left\langle v - Sv,\ Sv - v\right\rangle + \left\|Sv - v\right\|^2,$$

and hence, $0 \le -\left\|Sv - v\right\|^2$. This result indicates that $Sv = v$. Therefore, a nonexpansive mapping S with a fixed point is mean-demiclosed (2.5). □

The main theorems in this paper highlight quasi-nonexpansive and mean-demiclosed mappings. Note that this class of mappings contains more general classes of mappings than nonexpansive mappings. We refer to this point in Appendix A.

In the following sections, we assume that two nonlinear mappings have a common fixed point. It is known that if nonexpansive mappings are commutative and the domain of the mappings is closed, convex, and bounded, a common fixed point exists; see Browder [4]. For more broad classes of mappings including nonexpansive mappings, see Kondo [12] and papers cited therein.

3 Strong convergence by the hybrid method

In this section, we prove a strong convergence theorem for finding a common fixed point of two nonlinear mappings. The mappings are assumed to be quasi-nonexpansive and mean-demiclosed, but they are not required to be continuous or commutative. One important example of this class of mappings is a nonexpansive mapping with a fixed point. The basic element of the proof has been polished by many researchers [1, 6, 14, 21].

Theorem 3. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and $T : C \to C$ be quasi-nonexpansive and mean-demiclosed mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{z_n\}$ and $\{w_n\}$ be sequences in C. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$,*

$$\varliminf_{n\to\infty} a_n b_n > 0, \quad \text{and} \quad \varliminf_{n\to\infty} a_n c_n > 0. \tag{3.1}$$

Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned}
&x_1 = x \in C \text{ given,}\\
&y_n = a_n x_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n \in C,\\
&C_n = \{h \in C : \|y_n - h\| \le \|x_n - h\|\},\\
&Q_n = \{h \in C : \langle x - x_n,\ x_n - h\rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x \ \ \forall n \in \mathbb{N}.
\end{aligned}$$

Assume that

$$\|z_n - q\| \le \|x_n - q\| \quad \text{and} \quad \|w_n - q\| \le \|x_n - q\| \tag{3.2}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} = P_{F(S)\cap F(T)} x$.

Proof. Define

$$Z_n = \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n \quad \text{and} \quad W_n = \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n$$

for all $n \in \mathbb{N}$. The sequences $\{Z_n\}$ and $\{W_n\}$ are in C since C is convex. Using these notations, we can simply write as $y_n = a_n x_n + b_n Z_n + c_n W_n$. It can be verified that

$$\|Z_n - q\| \le \|z_n - q\| \quad \text{and} \quad \|W_n - q\| \le \|w_n - q\| \tag{3.3}$$

for all $n \in \mathbb{N}$ and $q \in F(S) \cap F(T)$. In fact, as S is quasi-nonexpansive and $q \in F(S)$, it holds that

$$\begin{aligned}
\|Z_n - q\| &= \left\|\frac{1}{n}\sum_{l=0}^{n-1} S^l z_n - q\right\| = \frac{1}{n}\left\|\sum_{l=0}^{n-1} S^l z_n - nq\right\| = \frac{1}{n}\left\|\sum_{l=0}^{n-1}\left(S^l z_n - q\right)\right\|\\
&\le \frac{1}{n}\sum_{l=0}^{n-1}\left\|S^l z_n - q\right\| \le \frac{1}{n}\sum_{l=0}^{n-1}\|z_n - q\| = \|z_n - q\|.
\end{aligned} \tag{3.4}$$

Similarly, $\|W_n - q\| \le \|w_n - q\|$ can be demonstrated since T is quasi-nonexpansive and $q \in F(T)$. Thus, (3.3) follows as claimed.

Note that once x_n and $y_n \in C$ are provided, the sets C_n and Q_n are defined and are closed and convex in C. We prove that

$$F(S) \cap F(T) \subset C_n \cap Q_n \quad \text{for all } n \in \mathbb{N}$$

using mathematical induction. (i) As $Q_1 = C$, it is obvious that $F(S) \cap F(T) \subset Q_1$. Choose $q \in F(S) \cap F(T)$ arbitrarily. It follows from (3.3) and (3.2) that

$$\begin{aligned} \|y_1 - q\| &= \|a_1 x_1 + b_1 Z_1 + c_1 W_1 - q\| \\ &= \|a_1 (x_1 - q) + b_1 (Z_1 - q) + c_1 (W_1 - q)\| \\ &\leq a_1 \|x_1 - q\| + b_1 \|Z_1 - q\| + c_1 \|W_1 - q\| \\ &\leq a_1 \|x_1 - q\| + b_1 \|z_1 - q\| + c_1 \|w_1 - q\| \\ &\leq a_1 \|x_1 - q\| + b_1 \|x_1 - q\| + c_1 \|x_1 - q\| = \|x_1 - q\|. \end{aligned} \tag{3.5}$$

This result shows that $F(S) \cap F(T) \subset C_1$ and hence, $F(S) \cap F(T) \subset C_1 \cap Q_1$. (ii) Assume that $F(S) \cap F(T) \subset C_k \cap Q_k$, where $k \in \mathbb{N}$. From the hypothesis $F(S) \cap F(T) \neq \emptyset$, we have $C_k \cap Q_k \neq \emptyset$. As $C_k \cap Q_k$ is a nonempty, closed, and convex subset of $C\,(\subset H)$, the metric projection $P_{C_k \cap Q_k}$ from H onto $C_k \cap Q_k$ is defined. Thus, $x_{k+1} = P_{C_k \cap Q_k} x$ is also defined. Furthermore, Z_{k+1}, W_{k+1}, $y_{k+1}\,(\in C)$, C_{k+1}, and $Q_{k+1}\,(\subset C)$ are defined as follows:

$$\begin{aligned} Z_{k+1} &= \frac{1}{k+1} \sum_{l=0}^{k} S^l z_{k+1}, \qquad W_{k+1} = \frac{1}{k+1} \sum_{l=0}^{k} T^l w_{k+1}, \\ y_{k+1} &= a_{k+1} x_{k+1} + b_{k+1} Z_{k+1} + c_{k+1} W_{k+1}, \\ C_{k+1} &= \{h \in C : \|y_{k+1} - h\| \leq \|x_{k+1} - h\|\}, \text{ and} \\ Q_{k+1} &= \{h \in C : \langle x - x_{k+1},\ x_{k+1} - h\rangle \geq 0\}. \end{aligned}$$

We demonstrate that $F(S) \cap F(T) \subset C_{k+1} \cap Q_{k+1}$. Let $q \in F(S) \cap F(T)$. From (3.3) and (3.2), it follows that

$$\begin{aligned} \|y_{k+1} - q\| &= \|a_{k+1} x_{k+1} + b_{k+1} Z_{k+1} + c_{k+1} W_{k+1} - q\| \\ &= \|a_{k+1} (x_{k+1} - q) + b_{k+1} (Z_{k+1} - q) + c_{k+1} (W_{k+1} - q)\| \\ &\leq a_{k+1} \|x_{k+1} - q\| + b_{k+1} \|Z_{k+1} - q\| + c_{k+1} \|W_{k+1} - q\| \\ &\leq a_{k+1} \|x_{k+1} - q\| + b_{k+1} \|z_{k+1} - q\| + c_{k+1} \|w_{k+1} - q\| \\ &\leq a_{k+1} \|x_{k+1} - q\| + b_{k+1} \|x_{k+1} - q\| + c_{k+1} \|x_{k+1} - q\| \\ &= \|x_{k+1} - q\|. \end{aligned} \tag{3.6}$$

This result implies that $q \in C_{k+1}$, and thus, we obtain $F(S) \cap F(T) \subset C_{k+1}$. Note that $x_{k+1} = P_{C_k \cap Q_k} x$ and $q \in F(S) \cap F(T) \subset C_k \cap Q_k$. Therefore, from (2.2), it follows that $\langle x - x_{k+1},\ x_{k+1} - q\rangle \geq 0$. This relationship implies that $q \in Q_{k+1}$. Thus, we obtain $F(S) \cap F(T) \subset C_{k+1} \cap Q_{k+1}$ as claimed. We have established that $F(S) \cap F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. Since $F(S) \cap F(T) \neq \emptyset$ is assumed, $C_n \cap Q_n \neq \emptyset$ for all $n \in \mathbb{N}$. Hence, the sequence $\{x_n\}$ is defined inductively. Note that letting $z_n = w_n = x_n$ for all $n \in \mathbb{N}$ demonstrates that the sequences $\{z_n\}$ and $\{w_n\}$, which satisfy (3.2), exist.

From the definition of Q_n, it holds that $x_n = P_{Q_n}x$ for all $n \in \mathbb{N}$. Then,

$$\|x - x_n\| \leq \|x - q\| \tag{3.7}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, since $x_n = P_{Q_n}x$ and $q \in F(S) \cap F(T) \subset C_n \cap Q_n \subset Q_n$, the inequality (3.7) holds. From (3.7), $\{x_n\}$ is bounded. From the assumption (3.2), $\{z_n\}$ and $\{w_n\}$ are also bounded.

As $x_n = P_{Q_n}x$ and $x_{n+1} = P_{C_n \cap Q_n}x \in Q_n$, it holds true that

$$\|x - x_n\| \leq \|x - x_{n+1}\| \tag{3.8}$$

for all $n \in \mathbb{N}$. This result indicates that the sequence $\{\|x - x_n\|\}$ in $\mathbb{R}$ is monotone increasing. As $\{x_n\}$ is bounded, so is $\{\|x - x_n\|\}$. Consequently, $\{\|x - x_n\|\}$ is convergent.

Next, we verify that

$$x_n - y_n \to 0. \tag{3.9}$$

As $x_n = P_{Q_n}x$ and $x_{n+1} = P_{C_n \cap Q_n}x \in Q_n$, we have from (2.3) that

$$\|x - x_n\|^2 + \|x_n - x_{n+1}\|^2 \leq \|x - x_{n+1}\|^2 \tag{3.10}$$

for all $n \in \mathbb{N}$. As $\{\|x - x_n\|\}$ is convergent, it follows that

$$x_n - x_{n+1} \to 0. \tag{3.11}$$

Furthermore, since $x_{n+1} = P_{C_n \cap Q_n}x \in C_n$, it holds that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. From (3.11), $y_n - x_{n+1} \to 0$. This information, together with (3.11), implies that $x_n - y_n \to 0$ as claimed.

Next, observe that

$$x_n - Z_n \to 0 \quad \text{and} \quad x_n - W_n \to 0. \tag{3.12}$$

Indeed, for $q \in F(S) \cap F(T)$, it follows from (2.1), (3.3), and (3.2) that

$$\begin{aligned}
\|y_n - q\|^2 &= \|a_n x_n + b_n Z_n + c_n W_n - q\|^2 \\
&= \|a_n (x_n - q) + b_n (Z_n - q) + c_n (W_n - q)\|^2 \\
&= a_n \|x_n - q\|^2 + b_n \|Z_n - q\|^2 + c_n \|W_n - q\|^2 \\
&- a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 \\
&\leq a_n \|x_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 \\
&- a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 \\
&\leq a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\
&- a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 \\
&= \|x_n - q\|^2 - a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 .
\end{aligned}$$

Using $b_n c_n \|Z_n - W_n\|^2 \geq 0$, we have

$$\begin{aligned}
&a_n b_n \|x_n - Z_n\|^2 + a_n c_n \|x_n - W_n\|^2 \\
&\leq \|x_n - q\|^2 - \|y_n - q\|^2 \\
&\leq \big(\|x_n - q\| + \|y_n - q\| \big) \left| \|x_n - q\| - \|y_n - q\| \right| \\
&\leq (\|x_n - q\| + \|y_n - q\|) \|x_n - y_n\| .
\end{aligned}$$

As $\{x_n\}$ is bounded, according to (3.9), $\{y_n\}$ is also bounded. Hence, we obtain from (3.9) and the assumption (3.1) regarding the coefficients a_n, b_n, and c_n that $x_n - Z_n \to 0$ and $x_n - W_n \to 0$ as claimed.

Our goal is to prove that $x_n \to \widehat{x} \left(= P_{F(S)\cap F(T)} x\right)$. Equivalently, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to \widehat{x}$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. As $\{x_{n_i}\}$ is bounded, there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup v$ for some $v \in H$. From (3.12), $Z_{n_j} \rightharpoonup v$ and $W_{n_j} \rightharpoonup v$. As S and T are mean-demiclosed (2.5), it follows that $v \in F(S) \cap F(T)$.

We show that $x_{n_j} \to v$. Using (3.7) for $q = v \in F(S) \cap F(T)$, we have

$$\begin{aligned}
\left\|x_{n_j} - v\right\|^2 &= \left\|x_{n_j} - x\right\|^2 + 2\left\langle x_{n_j} - x,\ x - v\right\rangle + \|x - v\|^2 \\
&\leq \|x - v\|^2 + 2\left\langle x_{n_j} - x,\ x - v\right\rangle + \|x - v\|^2 \\
&= 2\|x - v\|^2 + 2\left\langle x_{n_j} - x, x - v\right\rangle .
\end{aligned}$$

From $x_{n_j} \rightharpoonup v$, the right-most term tends to 0 as $j \to \infty$, that is,

$$2\|x - v\|^2 + 2\left\langle x_{n_j} - x,\ x - v\right\rangle \to 2\|x - v\|^2 + 2\langle v - x,\ x - v\rangle = 0.$$

Hence, we obtain $x_{n_j} \to v$ as claimed. Finally, we prove $v\Big(= \lim_{j\to\infty} x_{n_j}\Big) = \widehat{x}\left(= P_{F(S)\cap F(T)} x\right)$. As $\widehat{x} = P_{F(S)\cap F(T)} x$ and $v \in F(S) \cap F(T)$, it suffices to show that $\|x - v\| \leq \|x - \widehat{x}\|$. From (3.7) for $q = \widehat{x} \in F(S) \cap F(T)$, it holds true that $\left\|x - x_{n_j}\right\| \leq \|x - \widehat{x}\|$ for all $j \in \mathbb{N}$. As $x_{n_j} \to v$, we obtain $\|x - v\| \leq \|x - \widehat{x}\|$. This result indicates that $v = \widehat{x}$. We have proved that for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there is a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to \widehat{x} \,(= v)$. This completes the proof. □

From Theorem 3, the following result is obtained as a direct corollary:

Corollary 1 ([14]). Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and $T : C \to C$ be quasi-nonexpansive and mean-demiclosed mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $a, b \in (0, 1)$ with $a \leq b$ and let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers such that $0 < a \leq a_n$, b_n, $c_n \leq b < 1$ and $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in

C as follows:

$$
\begin{aligned}
&x_1 = x \in C \text{ given},\\
&y_n = a_n x_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l x_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l x_n \in C,\\
&C_n = \{h \in C : \|y_n - h\| \le \|x_n - h\|\},\\
&Q_n = \{h \in C : \langle x - x_n,\ x_n - h\rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} = P_{F(S)\cap F(T)} x$.

Proof. From the assumption that $0 < a \le a_n,\ b_n,\ c_n \le b < 1$, it holds that $\underline{\lim}_{n\to\infty} a_n b_n \ge a^2 > 0$ and $\underline{\lim}_{n\to\infty} a_n c_n \ge a^2 > 0$. Letting $z_n = w_n = x_n$ for all $n \in \mathbb{N}$ in Theorem 3, we obtain the desired result. □

4 Strong convergence by the shrinking projection method

In this section, we prove a strong convergence theorem for finding common fixed points for two nonlinear mappings using the shrinking projection method [28] together with the mean-valued iteration and the Ishikawa iteration. The basic element of the proof has been developed by many researchers; see, for instance, [6, 14].

To prove the main theorem in this section, the condition imposed on the mappings can be more relaxed than in the previous section. Let S be a self-mapping defined on C such that $F(S) \neq \emptyset$, where C is a nonempty, closed, and convex subset of a real Hilbert space H. Let $\{z_n\}$ be a bounded sequence in C and define $Z_n \equiv \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n\ (\in C)$ for each $n \in \mathbb{N}$. Consider the following property with respect to S:

$$
Z_{n_i} \to v \Longrightarrow v \in F(S), \tag{4.1}
$$

where $\{Z_{n_i}\}$ is a subsequence of $\{Z_n\}$. Mean-demiclosed mappings (2.5) sufficiently satisfy the property (4.1). According to Claim 1, nonexpansive mappings satisfy (4.1). It is known that more general types of mappings satisfy (4.1); see Appendix A in this paper or Kondo [14]. In this section, we focus on quasi-nonexpansive mappings that satisfy the property (4.1).

Theorem 4. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and $T : C \to C$ be quasi-nonexpansive mappings that satisfy $F(S) \cap F(T) \neq \emptyset$ and the property* (4.1)*. Let $\{z_n\}$ and $\{w_n\}$ be sequences in C. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$,*

$$
\underline{\lim}_{n\to\infty} a_n b_n > 0, \quad \textit{and} \quad \underline{\lim}_{n\to\infty} a_n c_n > 0. \tag{4.2}
$$

Let $\{u_n\}$ be a sequence in H such that $u_n \to u\,(\in H)$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned}
&x_1 = x \in C \text{ given},\\
&C_1 = C,\\
&y_n = a_n x_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n \in C,\\
&C_{n+1} = \{h \in C_n : \|y_n - h\| \le \|x_n - h\|\},\\
&x_{n+1} = P_{C_{n+1}} u_{n+1}
\end{aligned}$$

for all $n \in \mathbb{N}$. Assume that

$$\|z_n - q\| \le \|x_n - q\| \quad \text{and} \quad \|w_n - q\| \le \|x_n - q\| \tag{4.3}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{u}$ of $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.

Proof. We again use notations $Z_n = \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n$ and $W_n = \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n$. From the convexity of C, $\{Z_n\}$ and $\{W_n\}$ are in C. We have $y_n = a_n x_n + b_n Z_n + c_n W_n\,(\in C)$. It holds that

$$\|Z_n - q\| \le \|z_n - q\| \quad \text{and} \quad \|W_n - q\| \le \|w_n - q\| \tag{4.4}$$

for all $n \in \mathbb{N}$ and $q \in F(S) \cap F(T)$. This statement (4.4) can be proved in a similar manner to that used to prove (3.3) since S and T are quasi-nonexpansive.

Using mathematical induction, we can show that C_n is closed and convex as well as that $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$. (i) For $n = 1$, the results hold true since $C_1 = C$. (ii) Assume that C_k is closed and convex and $F(S) \cap F(T) \subset C_k$, where $k \in \mathbb{N}$. As $F(S) \cap F(T) \neq \emptyset$ and $F(S) \cap F(T) \subset C_k$, it holds that $C_k \neq \emptyset$. Consequently, the metric projection P_{C_k} exists, and x_k, Z_k, W_k, y_k, and C_{k+1} are defined as follows:

$$\begin{aligned}
&x_k = P_{C_k} u_k \in C_k \subset C,\\
&Z_k = \frac{1}{k}\sum_{l=0}^{k-1} S^l z_k, \quad W_k = \frac{1}{k}\sum_{l=0}^{k-1} T^l w_k,\\
&y_k = a_k x_k + b_k Z_k + c_k W_k \in C, \text{ and}\\
&C_{k+1} = \{h \in C_k : \|y_k - h\| \le \|x_k - h\|\} \subset C_k \subset C.
\end{aligned}$$

It can be easily ascertained that C_{k+1} is closed and convex since C_k is closed and convex. Let $q \in F(S) \cap F(T)$. We demonstrate that $q \in C_{k+1}$. From (4.4) and (4.3), it follows that

$$\begin{aligned}
\|y_k - q\| &= \|a_k x_k + b_k Z_k + c_k W_k - q\|\\
&= \left\|a_k (x_k - q) + b_k (Z_k - q) + c_k (W_k - q)\right\|\\
&\le a_k \|x_k - q\| + b_k \|Z_k - q\| + c_k \|W_k - q\|\\
&\le a_k \|x_k - q\| + b_k \|z_k - q\| + c_k \|w_k - q\|\\
&\le a_k \|x_k - q\| + b_k \|x_k - q\| + c_k \|x_k - q\| = \|x_k - q\|.
\end{aligned}$$

Thus, we obtain $q \in C_{k+1}$ as claimed. We have shown that C_n is a closed and convex subset of C and that $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$. From the assumption that $F(S) \cap F(T) \neq \emptyset$, we have $C_n \neq \emptyset$ for all $n \in \mathbb{N}$. Thus, the sequence $\{x_n\}$ in C is inductively defined. Note that letting $z_n = w_n = x_n$ for all $n \in \mathbb{N}$ shows that sequences $\{z_n\}$ and $\{w_n\}$ with the condition (4.3) exist.

Define $\overline{u}_n = P_{C_n} u \, (\in C_n)$. As $C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$, $\{\overline{u}_n\}$ is a sequence in C. Since $\overline{u}_n = P_{C_n} u$ and $F(S) \cap F(T) \subset C_n$, it holds true that

$$\|u - \overline{u}_n\| \leq \|u - q\| \tag{4.5}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This result shows that $\{\overline{u}_n\}$ is bounded. Furthermore, since $\overline{u}_n = P_{C_n} u$ and $\overline{u}_{n+1} = P_{C_{n+1}} u \in C_{n+1} \subset C_n$, it holds that

$$\|u - \overline{u}_n\| \leq \|u - \overline{u}_{n+1}\|$$

for all $n \in \mathbb{N}$. This result demonstrates that $\{\|u - \overline{u}_n\|\}$ is monotone increasing. As $\{\overline{u}_n\}$ is bounded, so is $\{\|u - \overline{u}_n\|\}$. Thus, the sequence $\{\|u - \overline{u}_n\|\}$ of real numbers is convergent.

Observe that $\{\overline{u}_n\}$ is convergent in C; in other words, there exists $\overline{u} \in C$ such that

$$\overline{u}_n \to \overline{u}. \tag{4.6}$$

Let $m, n \in \mathbb{N}$ with $m \geq n$. Since $\overline{u}_n = P_{C_n} u$ and $\overline{u}_m = P_{C_m} u \in C_m \subset C_n$, we have from (2.3) that

$$\|u - \overline{u}_n\|^2 + \|\overline{u}_n - \overline{u}_m\|^2 \leq \|u - \overline{u}_m\|^2 .$$

As $\{\|u - \overline{u}_n\|\}$ is convergent, it follows that $\overline{u}_n - \overline{u}_m \to 0$ as m and n tend to infinity. Equivalently, $\{\overline{u}_n\}$ is a Cauchy sequence in C. As C is closed in a real Hilbert space H, it is complete. Thus, there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ as claimed. Next, we claim that $\{x_n\}$ has the same limit, namely,

$$x_n \to \overline{u}. \tag{4.7}$$

Indeed, since the metric projection P_{C_n} is nonexpansive and $u_n \to u$ is assumed, it holds from (4.6) that

$$\begin{aligned} \|x_n - \overline{u}\| &\leq \|x_n - \overline{u}_n\| + \|\overline{u}_n - \overline{u}\| = \|P_{C_n} u_n - P_{C_n} u\| + \|\overline{u}_n - \overline{u}\| \\ &\leq \|u_n - u\| + \|\overline{u}_n - \overline{u}\| \to 0 \end{aligned}$$

as claimed. Consequently, $\{x_n\}$ is bounded. Our goal is to demonstrate that $\overline{u} = \widehat{u} \left(= P_{F(S) \cap F(T)} u\right)$ since $x_n \to \overline{u}$ is already proved.

We show that

$$x_n - y_n \to 0. \tag{4.8}$$

Indeed, as $\{x_n\}$ is convergent, $x_n - x_{n+1} \to 0$. From $x_{n+1} = P_{C_{n+1}} u_{n+1} \in C_{n+1}$, it follows that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Thus, we obtain

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \leq 2 \|x_n - x_{n+1}\| \to 0$$

as claimed. Since $\{x_n\}$ is bounded, $x_n - y_n \to 0$ implies that $\{y_n\}$ is also bounded. Next, we claim that

$$x_n - Z_n \to 0 \quad \text{and} \quad x_n - W_n \to 0. \tag{4.9}$$

Choose $q \in F(S) \cap F(T)$ arbitrarily. From (2.1), (4.4), and (4.3), it follows that

$$\begin{aligned}
&\|y_n - q\|^2 = \|a_n x_n + b_n Z_n + c_n W_n - q\|^2 \\
&= \left\|a_n (x_n - q) + b_n (Z_n - q) + c_n (W_n - q)\right\|^2 \\
&= a_n \|x_n - q\|^2 + b_n \|Z_n - q\|^2 + c_n \|W_n - q\|^2 \\
&\quad - a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 \\
&\le a_n \|x_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 \\
&\quad - a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 \\
&\le a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\
&\quad - a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 \\
&= \|x_n - q\|^2 - a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 .
\end{aligned}$$

As $b_n c_n \|Z_n - W_n\|^2 \ge 0$, we have

$$\begin{aligned}
a_n b_n &\|x_n - Z_n\|^2 + a_n c_n \|x_n - W_n\|^2 \\
&\le \|x_n - q\|^2 - \|y_n - q\|^2 \\
&\le (\|x_n - q\| + \|y_n - q\|) \left|\|x_n - q\| - \|y_n - q\|\right| \\
&\le (\|x_n - q\| + \|y_n - q\|) \|x_n - y_n\| .
\end{aligned}$$

As $\{x_n\}$ and $\{y_n\}$ are bounded, we obtain (4.9) from (4.8) and the assumption (4.2) regarding the coefficients a_n, b_n, and c_n. From (4.7) and (4.9), we have $Z_n \to \overline{u}$ and $W_n \to \overline{u}$. Since the mappings S and T satisfy the property (4.1), we obtain $\overline{u} \in F(S) \cap F(T)$.

Finally, we prove that $\overline{u}\,(= \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} x_n) = \widehat{u}\,\left(= P_{F(S)\cap F(T)} u\right)$. As $\overline{u} \in F(S) \cap F(T)$ and $\widehat{u} = P_{F(S)\cap F(T)} u$, it suffices to show that $\|u - \overline{u}\| \le \|u - \widehat{u}\|$. As $\widehat{u} \in F(S) \cap F(T)$, from (4.5), it holds that $\|u - \overline{u}_n\| \le \|u - \widehat{u}\|$. From (4.6), we obtain $\|u - \overline{u}\| \le \|u - \widehat{u}\|$. This result implies that $\overline{u} = \widehat{u}$. From (4.7), we obtain $x_n \to \widehat{u}\,(= \overline{u})$, which completes the proof. □

As in the previous section, the following result is derived:

Corollary 2 ([14]). Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and $T : C \to C$ be quasi-nonexpansive mappings that satisfy $F(S) \cap F(T) \neq \emptyset$ and the property (4.1). Let $a, b \in (0, 1)$ with $a \le b$ and let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers such that $0 < a \le a_n$, b_n, $c_n \le b < 1$ and $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$. Let $\{u_n\}$ be a sequence in

H such that $u_n \to u\,(\in H)$. Define a sequence $\{x_n\}$ in C as follows:

$$
\begin{aligned}
&x_1 = x \in C \text{ given},\\
&C_1 = C,\\
&y_n = a_n x_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n \in C,\\
&C_{n+1} = \{h \in C_n : \|y_n - h\| \le \|x_n - h\|\}, \text{ and}\\
&x_{n+1} = P_{C_{n+1}} u_{n+1}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{u}$ of $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.

Proof. The conditions $\underline{\lim}_{n\to\infty} a_n b_n > 0$ and $\underline{\lim}_{n\to\infty} a_n c_n > 0$ are satisfied from the hypothesis that $0 < a \le a_n,\ b_n,\ c_n \le b < 1$. Letting $z_n = w_n = x_n$ for all $n \in \mathbb{N}$ in Theorem 4, we obtain the desired result. □

5 Derivative results

In this section, we present a variety of strong convergence results that are derived from Theorems 3 and 4. To save space, we only refer to Theorem 3. First, the following result is obtained:

Theorem 5. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and $T : C \to C$ be quasi-nonexpansive and mean-demiclosed mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\nu_n\}$ be sequences of nonnegative real numbers such that $\lambda_n + \mu_n + \nu_n = 1$ for all $n \in \mathbb{N}$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$
\begin{aligned}
&x_1 = x \in C \text{ given},\\
&z_n = w_n = \lambda_n x_n + \mu_n S x_n + \nu_n T x_n,\\
&y_n = a_n x_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n \in C,\\
&C_n = \{h \in C : \|y_n - h\| \le \|x_n - h\|\},\\
&Q_n = \{h \in C : \langle x - x_n,\ x_n - h\rangle \ge 0\}, \text{ and}\\
&x_{n+1} = P_{C_n \cap Q_n} x
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} = P_{F(S)\cap F(T)} x$.

Proof. Given that $z_n = w_n$ is assumed, it suffices to demonstrate that

$$\|z_n - q\| \le \|x_n - q\|, \quad \forall q \in F(S) \cap F(T), \quad n \in \mathbb{N}.$$

As S and T are quasi-nonexpansive, it holds true that

$$\begin{aligned}\|z_n - q\| &= \|\lambda_n x_n + \mu_n S x_n + \nu_n T x_n - q\| \\ &= \|\lambda_n (x_n - q) + \mu_n (S x_n - q) + \nu_n (T x_n - q)\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|S x_n - q\| + \nu_n \|T x_n - q\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|x_n - q\| + \nu_n \|x_n - q\| = \|x_n - q\|.\end{aligned}$$

Therefore, the desired result follows from Theorem 3. □

Note that there are no required conditions on the nonnegative parameters $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\nu_n\}$ except for $\lambda_n + \mu_n + \nu_n = 1$. This theorem is based on Ishikawa iteration (1.2), Nakajo and Takahashi's hybrid method (1.4), and the mean-valued iteration (1.7). Clearly, the construction of z_n and w_n can be varied fairly freely. For example, the following iteration scheme sufficiently functions to derive a strong convergence:

$$z_n = \lambda_n x_n + \mu_n T x_n + \nu_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n, \tag{5.1}$$

$$w_n = \lambda'_n x_n + \mu'_n S x_n + \nu'_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n,$$

$$y_n = a_n x_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n \in C \quad \forall n \in \mathbb{N}.$$

In fact, we know that if $\|z_n - q\| \leq \|x_n - q\|$ and $\|w_n - q\| \leq \|x_n - q\|$ for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$, then a strong convergence follows from Theorem 3. It is easy to check that the iteration (5.1) meets these conditions. Note that in (5.1), z_n (resp. w_n) is affected only by the mapping T (resp. S) at least directly. Furthermore, the following "three-step iterative method" is obtained:

Theorem 6. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and $T : C \to C$ be quasi-nonexpansive and mean-demiclosed mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\nu_n\}$ be sequences of real numbers in the interval $[0, 1]$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned}&x_1 = x \in C \text{ given},\\ &w_n = \lambda_n x_n + (1 - \lambda_n) S x_n,\\ &z_n = \nu_n w_n + (1 - \nu_n) T w_n,\\ &y_n = a_n x_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n \in C,\\ &C_n = \{h \in C : \|y_n - h\| \leq \|x_n - h\|\},\\ &Q_n = \{h \in C : \langle x - x_n,\ x_n - h\rangle \geq 0\},\\ &x_{n+1} = P_{C_n \cap Q_n} x\end{aligned}$$

for all $n \in \mathbb{N}$*. Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{x}$ *of* $F(S) \cap F(T)$*, where* $\widehat{x} = P_{F(S)\cap F(T)}x$*.*

Proof. It suffices to demonstrate that $\|z_n - q\| \leq \|x_n - q\|$ for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. To that aim, first, we verify that $\|w_n - q\| \leq \|x_n - q\|$. In fact, as S is quasi-nonexpansive, it follows that

$$\begin{aligned}\|w_n - q\| &\leq \lambda_n \|x_n - q\| + (1 - \lambda_n) \|Sx_n - q\| \\ &\leq \lambda_n \|x_n - q\| + (1 - \lambda_n) \|x_n - q\| = \|x_n - q\| .\end{aligned}$$

Using this, we obtain

$$\begin{aligned}&\|z_n - q\| = \|\nu_n w_n + (1 - \nu_n) Tw_n - q\| \\ &\leq \nu_n \|w_n - q\| + (1 - \nu_n) \|Tw_n - q\| \\ &\leq \nu_n \|w_n - q\| + (1 - \nu_n) \|w_n - q\| \\ &= \|w_n - q\| \leq \|x_n - q\|\end{aligned}$$

since T is quasi-nonexpansive and $q \in F(T)$. This completes the proof. □

For the "three-step iterative method", see Noor [22], Phuengrattana and Suantai [23], and Chugh et al. [5]. Clearly, the "four-step iterative method"and other more general iterations are derived from Theorems 3 and 4. For other types of variations, see Kondo [13].

6 Application to the variational inequality problem

In this section, we present an application of a main result of this paper to the variational inequality problem. For detailed discussion about the following concepts and preliminary results, see Takahashi [26]. Let C be a nonempty, closed, and convex subset of a real Hilbert space H. A mapping $A : C \to H$ is called *K-Lipschitz continuous* if there exists $K > 0$ such that $\|Ax - Ay\| \leq K \|x - y\|$ for all $x, y \in C$. A mapping $A : C \to H$ is called *monotone* if

$$0 \leq \langle x - y,\ Ax - Ay\rangle \tag{6.1}$$

for all $x, y \in C$. A mapping $A : C \to H$ is called *strongly monotone* if

$$0 < \langle x - y,\ Ax - Ay\rangle \tag{6.2}$$

for $x, y \in C$ such that $x \neq y$. A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\alpha \|Ax - Ay\|^2 \leq \langle x - y,\ Ax - Ay\rangle \tag{6.3}$$

for all $x, y \in C$. A mapping $A : C \to H$ is called *η-strongly monotone* if there exists $\eta > 0$ such that

$$\eta \|x - y\|^2 \leq \langle x - y,\ Ax - Ay\rangle \tag{6.4}$$

for all $x, y \in C$. Concerning these classes of mappings, the following hold:

(A) An α-inverse strongly monotone mapping $A : C \to H$ is monotone and $(1/\alpha)$-Lipschitz continuous,

(B) An η-strongly monotone mapping $A : C \to H$ is strongly monotone,

(C) If a mapping A is K-Lipschitz continuous and η-strongly monotone, then A is $\left(\eta/K^2\right)$-inverse strongly monotone.

From (C), the class of α-inverse strongly monotone mappings is more general than Lipschitz continuous and strongly monotone mappings. In the following theorem (Theorem 7), we assume that a mapping A is an α-inverse strongly monotone.

A set of solutions to the variational inequality problem is denoted by

$$VI\,(C, A) = \{x \in C : \langle y - x,\ Ax\rangle \geq 0 \text{ for all } y \in C\}\,. \tag{6.5}$$

It is well-known that the variational inequality directly connects with optimization problems. For an illustration, suppose that the domain of the mapping A is the whole space H. Then, it is easy to verify that the set (6.5) of solution to the variational inequality problem coincides with the null point set of A, that is, $VI\,(H, A) = A^{-1}0$. If $H = \mathbb{R}$ and we interpret A as a derivative f' of a real-valued function f defined on $\mathbb{R}$, then $VI\,(\mathbb{R}, f')$ is the set of points $x \in \mathbb{R}$ that satisfies $f'(x) = 0$. In such a case, the assumption that $A\,(= f')$ is strongly monotone seems to be a bit strong.

The following facts are crucial to apply the fixed point theory to variational inequality problems:

(a) Let $A : C \to H$ be an α-inverse strongly monotone mapping. Then, for $\lambda \in [0, 2\alpha]$, $I - \lambda A$ is a nonexpansive mapping from C into H, where I is the identity mapping defined on C,

(b) Let $A : C \to H$ be an η-strongly monotone and K-Lipschitz continuous mapping. Then, for $\lambda \in \left(0, \frac{2\eta}{K^2}\right)$, $I - \lambda A$ is a contraction mapping from C into H, in other words, there exists $r \in (0, 1)$ such that $\|(I - \lambda A)\,x - (I - \lambda A)\,y\| \leq r\,\|x - y\|$ for all $x, y \in C$.

(c) It holds true that $VI\,(C, A) = F\,(P_C\,(I - \lambda A))$ for all $\lambda > 0$, where P_C is the metric projection from H onto C.

From (a) and (c), if C is nonempty, closed, convex, and bounded subset of H, then the set $VI\,(C, A)$ is nonempty, closed, and convex in C. In the case of (b), the set $VI\,(C, A)$ has only one element and well-known Picard iteration is effective to approximate the unique element of $VI\,(C, A)$. For case (b), see Yamada [32] and a recent contribution by Truong et al. [31]. In the next theorem, following Takahashi and Toyoda [29], we deal with the case (a).

Theorem 7. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping and let $A : C \to H$ be an α-inverse strongly monotone mapping. Suppose that $F\,(S) \cap VI\,(C, A) \neq \emptyset$. Let $\lambda \in (0, 2\alpha]$ and define $T = P_C\,(I - \lambda A)$. Let $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\nu_n\}$ be sequences of nonnegative real numbers such that $\lambda_n + \mu_n + \nu_n = 1$ for all $n \in \mathbb{N}$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers such that*

$a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned}
&x_1 = x \in C \text{ given},\\
&z_n = w_n = \lambda_n x_n + \mu_n S x_n + \nu_n T x_n,\\
&y_n = a_n x_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n \in C,\\
&C_n = \{h \in C : \|y_n - h\| \le \|x_n - h\|\},\\
&Q_n = \{h \in C : \langle x - x_n,\ x_n - h\rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x
\end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(S) \cap VI(C, A)$, where $\widehat{x} = P_{F(S)\cap VI(C,A)} x$.

Proof. From Claim 1, a nonexpansive mapping with a fixed point is quasi-nonexpansive and mean-demiclosed. From (a), $T : C \to C$ is nonexpansive. From (c), $F(T) = VI(C, A)$. Thus, we can apply Theorem 5 and obtain the desired result. □

This theorem demonstrates how to approximate a common element of a fixed point problem and a variational inequality problem.

7 Concluding remarks

In this paper, strong convergence theorems for finding common fixed points of two nonlinear mappings are proved. The methods are based on the iterative procedures introduced by Ishikawa, Nakajo and Takahashi, and Takahashi, Takeuchi, and Kubota, as well as the mean-valued method. Although two mappings are not assumed to be continuous nor commutative, they are required to be quasi-nonexpansive and mean-demiclosed. In addition to the nonexpansive mappings, the broad classes of mappings introduced in Appendix A are special cases of this class of mappings. Thus, the main theorems in this paper are applicable to those classes of mappings. As shown in Section 5, the main theorems in this paper can potentially yield many iteration procedures for approximating common fixed points of nonlinear mappings, which is one of the highlights of this paper. An application to the variational inequality problem is provided in Section 6. As a final remark, all results in this paper can be extended to any finite number of mappings.

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Appendix A

Quasi-nonexpansive and mean-demiclosed mappings

As mentioned in Sections 2 and 4, the class of quasi-nonexpansive and mean-demiclosed mappings includes more general types of mappings beyond nonexpansive mappings when they have fixed points. In this section, such classes of general types of mappings are introduced to show the wide range of applicability of the theorems demonstrated in this paper.

Let C be a nonempty subset of a real Hilbert space H. A mapping $S : C \to H$ is called

(i) *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$,

(ii) *nonspreading* [10] if

$$2\|Sx - Sy\|^2 \leq \|x - Sy\|^2 + \|Sx - y\|^2$$

for all $x, y \in C$.

It is well-known that nonexpansive mappings have a direct link with optimization problems; see Section 6. In Section 2, we have already shown that a nonexpansive mapping with a fixed point is quasi-nonexpansive and mean-demiclosed (Claim 1). Nonspreading mappings were also introduced because they are required by optimization problems; see Kohsaka and Takahashi [10]. Summing the inequalities in (i) and (ii), a concept of the following class of mappings is obtained. A mapping $S : C \to H$ is called

(iii) *hybrid* [27] if

$$3\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|x - Sy\|^2 + \|Sx - y\|^2$$

for all $x, y \in C$.

It is also known that the class of firmly nonexpansive mappings is a special case of (i)–(iii). These types of mappings are unified by the concept of *generalized hybrid mappings*. A mapping $S : C \to H$ is

(iv) *generalized hybrid* [9] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Sx - Sy\|^2 + (1 - \alpha)\|x - Sy\|^2 \leq \beta\|Sx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. If $\alpha = 1$ and $\beta = 0$, then a generalized hybrid mapping is nonexpansive, and therefore, the class of generalized hybrid mappings includes nonexpansive mappings. If $\alpha = 2$ and $\beta = 1$, a generalized hybrid mapping is nonspreading. A generalized hybrid mapping with $\alpha = 3/2$ and $\beta = 1/2$ is hybrid. Here, generalized hybrid mappings are further extended. A mapping $S : C \to H$ is said to be

(v) *normally generalized hybrid* [30] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$ and

$$\alpha \|Sx - Sy\|^2 + \beta \|x - Sy\|^2 + \gamma \|Sx - y\|^2 + \delta \|x - y\|^2 \leq 0$$

for all $x, y \in C$, where $\alpha + \beta > 0$, or $\alpha + \gamma > 0$. A normally generalized hybrid mapping with $\alpha + \beta = 1$ and $\gamma + \delta = -1$ is generalized hybrid, and hence, the class of normally generalized hybrid mappings contains all types of mappings (i)–(iv) as special cases.

The following types of mappings are generalizations of generalized and normally generalized hybrid mappings. A mapping $S : C \to C$ is called

(vi) 2-*generalized hybrid* [20] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1 \left\|S^2x - Sy\right\|^2 + \alpha_2 \|Sx - Sy\|^2 + (1 - \alpha_1 - \alpha_2) \|x - Sy\|^2 \\ &\leq \beta_1 \left\|S^2x - y\right\|^2 + \beta_2 \|Sx - y\|^2 + (1 - \beta_1 - \beta_2) \|x - y\|^2 \end{aligned} \tag{A.1}$$

for all $x, y \in C$,

(vii) *normally* 2-*generalized hybrid* [15] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{l=0}^{2} (\alpha_l + \beta_l) \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$, and

$$\begin{aligned} \alpha_2 \left\|S^2x - Sy\right\|^2 + \alpha_1 \|Sx - Sy\|^2 + \alpha_0 \|x - Sy\|^2 & \\ + \beta_2 \left\|S^2x - y\right\|^2 + \beta_1 \|Sx - y\|^2 + \beta_0 \|x - y\|^2 &\leq 0 \end{aligned} \tag{A.2}$$

for all $x, y \in C$. Clearly, if $\alpha_1 = \beta_1 = 0$ in (A.1), the mapping S is generalized hybrid. The class of normally 2-generalized hybrid mappings contains all types of mappings introduced here. Indeed, if $\alpha_2 + \alpha_1 + \alpha_0 = 1$ and $\beta_2 + \beta_1 + \beta_0 = -1$, then a normally 2-generalized hybrid mapping is 2-generalized hybrid. If $\alpha_2 = \beta_2 = 0$, then it is normally generalized hybrid. It is known that normally 2-generalized hybrid mappings are quasi-nonexpansive and mean-demiclosed if they have fixed points.

Claim 2 ([15]). *Let $S : C \to C$ be a normally 2-generalized hybrid mapping with $F(S) \neq \emptyset$, where C is a nonempty subset of H. Then, S is quasi-nonexpansive.*

Claim 3 ([16]). *Let $S : C \to C$ be a normally 2-generalized hybrid mapping with $F(S) \neq \emptyset$, where C is a nonempty, closed, and convex subset of H. Then, S is mean-demiclosed.*

For proofs of Claims 2 and 3, see also Kondo [11] and [14], respectively. As a normally 2-generalized hybrid mapping is mean-demiclosed, it satisfies the property (4.1), which is required for Theorem 4. As all classes of mappings (i)–(vi) are special cases of normally 2-generalized hybrid mappings, they are within the class of mappings targeted in this paper.

To provide an illustration, two examples of normally 2-generalized hybrid mappings are presented below; these examples are also provided in Kondo [11]. Let $\alpha_2 = \alpha$, $\beta_2 = -\beta$ with $0 < \beta < \alpha$ and let all the other coefficients be 0 in (A.2). Then, we have

$$\alpha \left\|S^2x - Sy\right\|^2 \leq \beta \left\|S^2x - y\right\|^2. \tag{A.3}$$

Example 1. [11]. Let $H = C = \mathbb{R}$, and define $S : \mathbb{R} \to \mathbb{R}$ as follows:

$$Sx = \begin{cases} 0, & \text{if } x \leq \sqrt{\alpha}, \\ \sqrt{\beta}, & \text{if } x > \sqrt{\alpha}. \end{cases}$$

We verify that the mapping S satisfies (A.3). In fact, from the hypothesis $\beta < \alpha$, it holds that $S^2x = 0$ for all $x \in \mathbb{R}$. Thus,

$$(A.3) \iff \alpha\left(Sy\right)^2 \leq \beta y^2 \iff |Sy| \leq \sqrt{\frac{\beta}{\alpha}}\,|y|\,. \tag{A.4}$$

(i) If $y \leq \sqrt{\alpha}$, then $|Sy| = 0$. Consequently, (A.4) holds. (ii) Assume that $y > \sqrt{\alpha}$. Then, the LHS of (A.4) is $|Sy| = \sqrt{\beta}$. The RHS is greater than $\sqrt{\frac{\beta}{\alpha}}\sqrt{\alpha} = \sqrt{\beta}$. Therefore, (A.4) holds true. This result indicates that S is normally 2-generalized hybrid. □

Example 2. [11]. Set $C = H$ and let P_U be the metric projection from H onto the unit sphere U. Define $S : H \to H$ as follows:

$$Sx = \begin{cases} P_U x, & \text{if } \sqrt{\frac{\alpha}{\beta}} < \|x\|\,, \\ 0, & \text{if } \|x\| \leq \sqrt{\frac{\alpha}{\beta}}. \end{cases}$$

Since $\beta < \alpha$, $S^2x = 0$ for all $x \in H$. Consequently,

$$(A.3) \iff \|Sy\| \leq \sqrt{\frac{\beta}{\alpha}}\,\|y\|\,. \tag{A.5}$$

If $\|y\| \leq \sqrt{\alpha/\beta}$, then $\|Sy\| = 0$, which means that (A.5) holds. If $\|y\| > \sqrt{\alpha/\beta}$, then the LHS of (A.5) is $LHS = \|P_U y\| = 1$. The RHS is greater than $\sqrt{\frac{\beta}{\alpha}}\sqrt{\frac{\alpha}{\beta}} = 1$. Thus, (A.5) holds true. □

As the mappings in Examples 1 and 2 are normally 2-generalized hybrid with fixed points, they are quasi-nonexpansive and mean-demiclosed. Hence, they are in the class of mappings addressed in this paper although they are not continuous.