

# On the Consistency and Convergence of Classical Richardson Extrapolation as Applied to Explicit One-Step Methods

Teshome Bayleyegn<sup>a</sup>, István Faragó<sup>a,b,c</sup> and Ágnes Havasi<sup>a,b</sup>

<sup>a</sup>*Applied Analysis and Computational Mathematics, Eötvös Loránd University Pázmány Péter s. 1/C, 1117 Budapest, Hungary*

<sup>b</sup>*ELKH-ELTE Numerical Analysis and Large Networks Research Group Pázmány Péter s. 1/C, 1117 Budapest, Hungary*

<sup>c</sup>*Institute of Mathematics, Budapest University of Technology and Economics Egrý J. u. 1., 1111 Budapest, Hungary*

E-mail: sbayleyegn130@gmail.com

E-mail: faragois@gmail.com

E-mail(*corresp.*): agnes.havasi@ttk.elte.hu

Received January 11, 2022; revised November 15, 2022; accepted November 15, 2022

**Abstract.** The consistency of the classical Richardson extrapolation (CRE), a simple and robust computational device, is analysed for the case where the underlying method is an explicit one-step numerical method for ordinary differential equations with order of consistency one or two. It is shown in the classical framework that the CRE increases the order of consistency by one. The convergence of the method is proved by the assumption that the time-stepping operator of the base method has the Lipschitz property in its second argument.

**Keywords:** consistency, convergence, explicit method, Richardson extrapolation, Taylor series expansion.

**AMS Subject Classification:** 65L06; 65L20; 65M12.

## 1 Introduction

Ordinary and partial differential equations are often at the core of solving environmental, engineering etc. problems. Numerical simulations, relying on

numerical methods for solving differential equations [2, 3, 7, 8, 9, 10] play an indispensable role in understanding the mechanisms of fluid flow, heat transfer, electric circuits, etc.

In certain model problems the accuracy of the numerical solution obtained by a chosen numerical method is not sufficient. In this case we can either decrease the time-step size, and perform the integration with the smaller time-step, or choose a higher-order numerical method, which is, as a rule, more costly. In both cases our previous calculations are thrown away. Another approach is the classical Richardson extrapolation (CRE) [11, 12, 14, 15, 16], where a suitable linear combination of numerical solutions obtained by two different time-step sizes with the same base method (BM) are calculated. An attractive property of this procedure is that its implementation requires relatively simple modifications in the computer code of a BM whose code is available.

Numerous experiments have shown that the CRE increases the accuracy of the applied BM by one [15], and for special (symmetric) numerical methods [5, 6], where the global error has an asymptotic expansion in even powers of the time-step, the order increases by two. In these experiments presented in [15], the maximal error committed during the time integration or the error committed at a given time point is investigated as the time step size is decreased, which serves as an experimental confirmation that the numerical solution is convergent to the exact solution. Convergence in this sense is a major requirement for a numerical time integration method, and the higher the convergence order, the smaller time step can be used for achieving a prescribed accuracy. According to the classical theory of numerical analysis, when a one-step numerical method is applied, the Lipschitz property of the time-stepping operator implies convergence in case of consistency, and the order of the convergence is equal to the order of consistency. Therefore, the knowledge of the consistency and its order is of crucial importance to study the convergence. Obviously, a higher order consistency is favourable, especially when accuracy requirements are higher.

In [7] the order of consistency is shown to increase to  $p + 1$  when a  $p$ -th order Runge–Kutta base method is combined with the CRE, by considering the equivalent autonomous problem. In this paper we use a different approach. We do not consider the equivalent autonomous problem, but a non-autonomous problem directly. Moreover, any explicit one-step method can serve as a BM in our case. Our analysis is based on a classical Taylor series expansion. We restrict ourselves to the cases where the BM has either order one or order two, since in most engineering applications second or third order of consistency is required.

The structure of the paper is as follows. In Section 2, the conditions of order  $p$  of a general explicit one-step method are derived for a non-autonomous Cauchy problem. In Section 3, we prove that the order of consistency is increased by one when the chosen BM of consistency order  $p = 1$  or  $p = 2$  is combined with the CRE. Then, in Section 4, the Lipschitz property of the time-stepping function of any explicit one-step BM of consistency order  $p = 1$  or  $p = 2$  combined with the CRE is proven, provided the BM satisfies a Lipschitz property. This implies that in this case the combined scheme is convergent,

and the order of convergence is equal to the order of consistency. Finally, our theoretical results are illustrated with numerical experiments.

## 2 Consistency of explicit one-step methods

Consider the Cauchy problem

$$u'(t) = f(t, u(t)), \quad u(0) = u_0, \quad (2.1)$$

where  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  (scalar case). Later on we always assume that this problem has a unique, sufficiently smooth solution  $u(t)$ . Assume that (2.1) is to be solved by the general explicit one-step method

$$y_n = y_{n-1} + h\Phi(t_{n-1}, y_{n-1}, h), \quad (2.2)$$

where  $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a smooth increment function. As known, the truncation error of this method [13] is defined as

$$T_n(h) = \frac{u(t_n) - u(t_{n-1})}{h} - \Phi(t_{n-1}, u(t_{n-1}), h),$$

and the method (2.2) is said to be  $p$ -th order consistent when  $T_n(h) = \mathcal{O}(h^p)$ .

First, we examine the conditions of first-order consistency. Applying Taylor series expansion for the exact solution around  $t_{n-1}$  and for the function  $\Phi$  around zero, we get

$$T_n(h) = u'(t_{n-1}) - \Phi(t_{n-1}, u(t_{n-1}), 0) + \mathcal{O}(h).$$

Hence,  $T_n(h) = \mathcal{O}(h)$  if and only if  $u'(t_{n-1}) = \Phi(t_{n-1}, u(t_{n-1}), 0)$ . In view of the differential equation, this implies the requirement  $\Phi(t_{n-1}, u(t_{n-1}), 0) = f(t_{n-1}, u(t_{n-1}))$ . Hence, the first order consistency is achieved if and only if the condition

$$\Phi(t, u, 0) = f(t, u) \quad (2.3)$$

holds for any  $(t, u) \in \mathbb{R}^+ \times \mathbb{R}$ . This result can be found in [9], p. 152, however, the order is not investigated there. However, in a similar way as above, the necessary and sufficient conditions of  $p$ -th order consistency can be given as

$$\frac{1}{k}u^{(k)}(t_{n-1}) = \partial_3^{k-1}\Phi(t_{n-1}, u(t_{n-1}), 0), \quad k = 1, \dots, p.$$

Particularly, the method is second order consistent if, in addition to (2.3), the condition

$$\partial_3\Phi(t, u, 0) = \frac{1}{2}[\partial_1 f(t, u) + \partial_2 f(t, u) \cdot f(t, u)] = \frac{1}{2}u''(t) \quad (2.4)$$

holds.

*Remark 1.* Conditions of  $p$ -th order consistency are derived in [1], but for autonomous systems, only. Namely, for the autonomous equation

$$y' = F(y), \quad (2.5)$$

where  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , the numerical method

$$y_{n+1}^h = \Phi^h(y_n^h) \tag{2.6}$$

is considered. Let

$$\Psi^h(y) = \frac{1}{h} (\Phi^h(y_n) - y)$$

be the increment function. The numerical method (2.6) is consistent with (2.5) iff

$$\Psi^0(y) = F(y).$$

Moreover, the method (2.6) is consistent with (2.5) at order  $p$ , if  $F \in C^p$  and

$$\frac{\partial^k \Psi^h(y)}{\partial h^k} = \frac{1}{k+1} F^{[k]}(y), \quad k = 0, 1, \dots, p,$$

where  $F^{[0]}, \dots, F^{[p-1]}$  are functions of the type  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  defined recursively, and the formulated conditions are rather complicated to check. Our approach is for the scalar non-autonomous case, which results in a condition that is simple and easy to check.

*Example 1.* For the explicit Euler method (EE), we have  $\Phi_{EE}(t, u, h) = f(t, u)$ , so (2.3) trivially holds, while (2.4) does not, so this method has first order.

*Example 2.* For the modified Euler method (MEE) the increment function reads

$$\Phi_{MEE}(t, u, h) = f\left(t + \frac{h}{2}, u + \frac{h}{2}f(t, u)\right),$$

so (2.3) trivially holds, and so does (2.4), since

$$\partial_3 \Phi(t, u, h) = \frac{1}{2} \partial_1 f\left(t + \frac{h}{2}, u + \frac{h}{2}f(t, u)\right) + \frac{1}{2} f(t, u) \partial_2 f\left(t + \frac{h}{2}, u + \frac{h}{2}f(t, u)\right),$$

and the substitution  $h = 0$  directly yields (2.4). Therefore, the method has (at least) second order.

### 3 Order conditions for the CRE

Assume that some BM is applied to solve the Cauchy problem (2.1). We divide  $[0, T]$  into  $N$  sub-intervals of length  $h$ , and define two meshes on  $[0, T]$ :

$$\Omega_h^0 := \{t_i = ih : i = 0, 1, \dots, N\}, \tag{3.1}$$

$$\Omega_h^1 := \{t_j = j\frac{h}{2} : j = 0, 1, \dots, 2N\}. \tag{3.2}$$

Note that  $\Omega_h^0 \subset \Omega_h^1$ .

We will only deal with the active Richardson extrapolation, which is defined as follows. Assume that the numerical solution has been calculated at the time level  $t_{n-1}$  of the coarse mesh  $\Omega_h^0$ , and denote it by  $y_{n-1}$ . Take a step of length  $h$  by using the BM, the resulting numerical solution will be denoted as  $z_n$ .

Take also two steps of step size  $h/2$  with the BM from  $y_{n-1}$ , and denote the obtained numerical solution by  $w_n$  (belonging also to time  $t_n$  of the mesh  $\Omega_h^0$ ). The CRE method is defined as

$$y_n := \frac{2^p w_n - z_n}{2^p - 1}, n = 1, 2, \dots, N. \quad (3.3)$$

*Remark 2.* The other variant of the CRE, not considered in this paper, is called passive Richardson extrapolation, where the problem is solved independently on both meshes, (3.1) and (3.2), and then the two solutions are combined linearly with the same weights as those under (3.3) in each point of the coarse mesh (3.1).

**Case  $p = 1$ :**

Assume that we have chosen an explicit one-step base method, i.e., a BM of the form (2.2) with time-stepping function  $\Phi_b$ , which is first order consistent, i.e., (2.3) holds. We show that its combination with CRE yields a second-order consistent method. To this aim, we first derive the increment function  $\Phi_{b,\text{CRE}}$  of the combined method.

1. Take one step of length  $h$ :  $z_n = y_{n-1} + h\Phi_b(t_{n-1}, y_{n-1}, h)$ .

2. Take two steps of length  $h/2$ : by using the notation  $t_{n-\frac{1}{2}} := t_{n-1} + \frac{h}{2}$  we can write

$$\begin{aligned} w_n &= y_{n-1} + 0.5h\Phi_b(t_{n-1}, y_{n-1}, 0.5h) \\ &\quad + 0.5h\Phi_b\left(t_{n-\frac{1}{2}}, y_{n-1} + 0.5h\Phi_b(t_{n-1}, y_{n-1}, 0.5h), 0.5h\right). \end{aligned}$$

3. Combine the two numerical solutions:  $y_n = 2w_n - z_n$ .

So, according to (2.1) we have

$$\begin{aligned} \Phi_{b,\text{CRE}}(t_{n-1}, y_{n-1}, h) &= \Phi_b(t_{n-1}, y_{n-1}, 0.5h) \\ &\quad + \Phi_b\left(t_{n-\frac{1}{2}}, y_{n-1} + 0.5h\Phi_b(t_{n-1}, y_{n-1}, 0.5h), 0.5h\right) - \Phi_b(t_{n-1}, y_{n-1}, h). \end{aligned}$$

Hence, the three-variable function determining the combined method reads as

$$\begin{aligned} \Phi_{b,\text{CRE}}(t, u, h) &= \Phi_b(t, u, 0.5h) \\ &\quad + \Phi_b\left(t + 0.5h, u + 0.5h\Phi_b(t, u, 0.5h), 0.5h\right) - \Phi_b(t, u, h). \end{aligned} \quad (3.4)$$

Let us check the conditions of order 2 under (2.3)–(2.4) for the increment function  $\Phi_{b,\text{CRE}}(t, u, h)$  defined by (3.4). Since the base method  $\Phi_b$  has first order, therefore (2.3) holds due to the relation

$$\Phi_{b,\text{CRE}}(t, u, 0) = \Phi_b(t, u, 0) = f(t, u).$$

For the condition (2.4) we have

$$\begin{aligned} \partial_3 \Phi_{b,\text{CRE}}(t, u, h) &= \frac{1}{2} \partial_3 \Phi_b\left(t, u, \frac{h}{2}\right) + \frac{1}{2} \partial_1 \Phi_b\left(t + \frac{h}{2}, u + \frac{h}{2} \Phi_b\left(t, u, \frac{h}{2}\right), \frac{h}{2}\right) \\ &\quad + \partial_2 \Phi_b\left(t + \frac{h}{2}, u + \frac{h}{2} \Phi_b\left(t, u, \frac{h}{2}\right), \frac{h}{2}\right) \cdot \left(\frac{1}{2} \Phi_b\left(t, u, \frac{h}{2}\right) + \frac{h}{4} \partial_3 \Phi_b\left(t, u, \frac{h}{2}\right)\right) \\ &\quad + \frac{1}{2} \partial_3 \Phi_b\left(t + \frac{h}{2}, u + \frac{h}{2} \Phi_b\left(t, u, \frac{h}{2}\right), \frac{h}{2}\right) - \partial_3 \Phi_b(t, u, h). \end{aligned}$$

Taking this function at  $h = 0$  yields

$$\partial_3\Phi_{b,\text{CRE}}(t, u, 0) = 0.5\partial_1\Phi_b(t, u, 0) + \partial_2\Phi_b(t, u, 0)0.5(\Phi_b(t, u, 0)).$$

Here,  $\Phi_b(t, u, 0) = f(t, u)$ ,  $\partial_1\Phi_b(t, u, 0) = \partial_1f(t, u)$  and  $\partial_2\Phi_b(t, u, 0) = \partial_2f(t, u)$ , therefore (2.4) is valid.

**Case  $p = 2$ :**

Assume that the base method has order 2. Then the numerical solution obtained by CRE reads

$$y_n = \frac{4}{3}w_n - \frac{1}{3}z_n = y_{n-1} - \frac{1}{3}h\Phi_b(t_{n-1}, y_{n-1}, h) + \frac{2}{3}h\Phi_b\left(t_{n-1}, y_{n-1}, \frac{h}{2}\right) + \frac{2}{3}h\Phi_b\left(t_{n-1} + \frac{h}{2}, y_{n-1} + \frac{h}{2}\Phi_b\left(t_{n-1}, y_{n-1}, \frac{h}{2}\right), \frac{h}{2}\right).$$

Therefore, the function of the combined method is

$$\begin{aligned} \Phi_{b,\text{CRE}}(t, u, h) &= \frac{2}{3}\Phi_b\left(t, u, \frac{h}{2}\right) + \frac{2}{3}\Phi_b\left(t + \frac{h}{2}, u + \frac{h}{2}\Phi_b\left(t, u, \frac{h}{2}\right), \frac{h}{2}\right) \\ &\quad - \frac{1}{3}\Phi_b(t, u, h). \end{aligned} \tag{3.5}$$

Assuming that the base method satisfies conditions (2.3)–(2.4), we need to show that the following three conditions are satisfied:

$$\Phi_{b,\text{CRE}}(t, u, 0) = f(t, u), \tag{3.6}$$

$$\partial_3\Phi_{b,\text{CRE}}(t, u, 0) = 0.5[\partial_1f + \partial_2f \cdot f](t, u), \tag{3.7}$$

$$\begin{aligned} \partial_{33}\Phi_{b,\text{CRE}}(t, u, 0) &= \frac{1}{3}[\partial_1^2f + 2\partial_1\partial_2f \cdot f + \partial_1f \cdot \partial_2f + \partial_2^2f \cdot f^2 \\ &\quad + (\partial_2f)^2 \cdot f](t, u). \end{aligned} \tag{3.8}$$

In the sequel, when no argument is written for  $f$ , the functions are to be understood at  $(t, u)$ . Checking condition (3.6):

$$\Phi_{b,\text{CRE}}(t, u, 0) = \frac{2}{3}\Phi_b(t, u, 0) + \frac{2}{3}\Phi_b(t, u, 0) - \frac{1}{3}\Phi_b(t, u, 0) = \Phi_b(t, u, 0) = f(t, u).$$

Checking condition (3.7):

$$\begin{aligned} \partial_3\Phi_{b,\text{CRE}}(t, u, h) &= \frac{2}{3} \cdot \frac{1}{2}\partial_3\Phi_b\left(t, u, \frac{h}{2}\right) + \frac{2}{3} \cdot \frac{1}{2}\partial_1\Phi_b\left(t + \frac{h}{2}, u + \frac{h}{2}\Phi_b\left(t, u, \frac{h}{2}\right), \frac{h}{2}\right) \\ &\quad + \frac{2}{3}\partial_2\Phi_b\left(t + \frac{h}{2}, u + \frac{h}{2}\Phi_b\left(t, u, \frac{h}{2}\right), \frac{h}{2}\right) \cdot \left(\frac{1}{2}\Phi_b\left(t, u, \frac{h}{2}\right) + \frac{h}{4}\partial_3\Phi_b\left(t, u, \frac{h}{2}\right)\right) \\ &\quad + \frac{2}{3} \cdot \frac{1}{2}\partial_3\Phi_b\left(t + \frac{h}{2}, u + \frac{h}{2}\Phi_b\left(t, u, \frac{h}{2}\right), \frac{h}{2}\right) - \frac{1}{3}\partial_3\Phi_b(t, u, h). \end{aligned}$$

Substituting  $h = 0$  yields

$$\begin{aligned} \partial_3\Phi_{b,\text{CRE}}(t, u, 0) &= \frac{1}{3}\partial_3\Phi_b(t, u, 0) + \frac{1}{3}\partial_1\Phi_b(t, u, 0) + \frac{2}{3}\partial_2\Phi_b(t, u, 0) \\ &\quad \cdot \frac{1}{2}\Phi_b(t, u, 0) + \frac{1}{3}\partial_3\Phi_b(t, u, 0) - \frac{1}{3}\partial_3\Phi_b(t, u, 0) \\ &= \frac{1}{3} \cdot \frac{1}{2}(\partial_1f + \partial_2f \cdot f) + \frac{1}{3}\partial_1f + \frac{1}{3}\partial_2f \cdot f = \frac{1}{2}[\partial_1f + \partial_2f \cdot f]. \end{aligned}$$

Checking condition (3.8): a simple, but tedious calculation shows that the function  $\partial_{33}\Phi_{b,\text{CRE}}(t, u, h)$  at  $h = 0$  is equal to

$$\begin{aligned} \partial_{33}\Phi_{b,\text{CRE}}(t, u, 0) &= \frac{1}{6}\partial_3^2\Phi_b(t, u, 0) - \frac{1}{3}\partial_3^2\Phi_b(t, u, 0) + \frac{1}{6}\partial_3^2\Phi_b(t, u, 0) \\ &+ \frac{1}{6}\partial_1^2\Phi_b(t, u, 0) + \frac{1}{3}\partial_2\partial_1\Phi_b(t, u, 0) \cdot \frac{1}{2}\Phi_b(t, u, 0) + \frac{1}{6}\partial_3\partial_1\Phi_b(t, u, 0) \\ &+ \left\{ \frac{1}{3}\partial_1\partial_2\Phi_b(t, u, 0) + \frac{2}{3}\partial_2^2\Phi_b(t, u, 0)\frac{1}{2}\Phi_b(t, u, 0) + \frac{1}{3}\partial_3\partial_2\Phi_b(t, u, 0) \right\} \\ &\cdot \frac{1}{2}\Phi_b(t, u, 0) + \frac{2}{3}\partial_2\Phi_b(t, u, 0) \cdot \left( \frac{1}{4}\partial_3\Phi_b(t, u, 0) \cdot 2 \right) + \frac{1}{6}\partial_1\partial_3\Phi_b(t, u, 0) \\ &+ \frac{1}{3}\partial_2\partial_3\Phi_b(t, u, 0)\frac{1}{2}\Phi_b(t, u, 0). \end{aligned}$$

By using the relationships  $\partial_i\Phi_b(t, u, 0) = \partial_i f(t, u)$  for  $i = 1, 2$ , and  $\partial_3\Phi_b(t, u, 0) = \frac{1}{2}[\partial_1 f + \partial_2 f \cdot f]$ , and always assuming sufficient smoothness of  $\Phi_b$  and  $f$  for the interchangeability of the differentiation with respect to any two variables, we obtain that

$$\begin{aligned} \partial_{33}\Phi_{b,\text{CRE}}(t, u, 0) &= \frac{1}{6}\partial_1^2 f + \frac{1}{3}\partial_2\partial_1 f \frac{1}{2}f + \frac{1}{6}\partial_1 \left( \frac{1}{2}[\partial_1 f + \partial_2 f \cdot f] \right) + \frac{1}{6}\partial_1\partial_2 f \cdot f \\ &+ \frac{1}{6}\partial_2^2 f \cdot f^2 + \frac{1}{6}\partial_2 \left( \frac{1}{2}[\partial_1 f + \partial_2 f \cdot f] \right) \cdot f + \frac{1}{6}\partial_1 \left( \frac{1}{2}[\partial_1 f + \partial_2 f \cdot f] \right) \\ &+ \frac{1}{6}\partial_2 \left( \frac{1}{2}[\partial_1 f + \partial_2 f \cdot f] \right) \cdot f + \frac{1}{3}\partial_2 f \left( \frac{1}{2}[\partial_1 f + \partial_2 f \cdot f] \right) \\ &= \frac{1}{3}[\partial_1^2 f + 2\partial_1\partial_2 f \cdot f + \partial_1 f \cdot \partial_2 f + \partial_2^2 f \cdot f^2 + (\partial_2 f)^2 \cdot f]. \end{aligned}$$

Here, we used the equalities  $\partial_1[\partial_1 f + \partial_2 f \cdot f] = \partial_1^2 f + \partial_1\partial_2 f \cdot f + \partial_2 f \cdot \partial_1 f$  and  $\partial_2[\partial_1 f + \partial_2 f \cdot f] = \partial_2\partial_1 f + \partial_2^2 f \cdot f + (\partial_2 f)^2$ .

## 4 Convergence analysis

We refer the following theorem from [13].

**Theorem 1.** *Consider a one-step method (2.2) with order of consistency  $p$ , where  $\Phi$  satisfies the Lipschitz condition with respect to its second argument uniformly in  $t$ , i.e., there exist constants  $L_\Phi$  and  $h_0$  such that for  $0 \leq h \leq h_0$*

$$|\Phi(t, y; h) - \Phi(t, z; h)| \leq L_\Phi |y - z| \quad \forall (t, y), (t, z) \in \tilde{R} := [0, T] \times \mathbb{R}.$$

Let  $e_n := u(t_n) - y_n$  be the global error at the  $n$ -th time step. Then, in case  $e_0 = \mathcal{O}(h^p)$ , we have that  $|e_n| \leq \mathcal{O}(h^p)$ , so the method is convergent of order  $p$ .

Our aim is to show the Lipschitz property for the time-stepping operator of the combined method. We consider the cases  $p = 1$  and  $p = 2$  separately. In both cases we will assume that the BM satisfies the Lipschitz condition

$$|\Phi_b(t, u, h) - \Phi_b(t, v, h)| \leq L_b |u - v|. \quad (4.1)$$

We remark that this has been shown for the ERK methods if the right-hand side function  $f$  satisfies the Lipschitz property in its second argument [4].

### 4.1 The Lipschitz property of $\Phi_{b,CRE}$ for $p = 1$

Let us show now that  $\Phi_{b,CRE}$  is Lipschitz in its second variable if  $\Phi_b$  satisfies (4.1) with Lipschitz constant  $L_{b,1}$ .

We need to show that there exists  $L_{CRE,1} > 0$  constant such that

$$\begin{aligned}
 |\Phi_{b,CRE}(t, u, h) - \Phi_{b,CRE}(t, v, h)| &\leq L_{CRE,1}|u - v|. \tag{4.2} \\
 |\Phi_{b,CRE}(t, u, h) - \Phi_{b,CRE}(t, v, h)| &\leq |\Phi_b(t, u, h/2) - \Phi_b(t, v, h/2)| \\
 &+ |\Phi_b(t + h/2, u + h/2\Phi_b(t, u, h/2) - \Phi_b(t + h/2, v + h/2\Phi_b(t, v, h/2), h/2)| \\
 &+ |\Phi_b(t, u, h) - \Phi_b(t, v, h)| \\
 &\leq L_{b,1}|u - v| + L_{b,1}\left|u + \frac{h}{2}\Phi_b\left(t, u, \frac{h}{2}\right) - v - \frac{h}{2}\Phi_b\left(t, v, \frac{h}{2}\right)\right| + \leq L_{b,1}|u - v| \\
 &\leq 2L_{b,1}|u - v| + L_{b,1}|u - v| + L_{b,1}\frac{h}{2}\left|\Phi_b\left(t, u, \frac{h}{2}\right) - \Phi_b\left(t, v, \frac{h}{2}\right)\right| \\
 &\leq L_{b,1}(3 + h/2)|u - v| \leq L_{b,1}(3 + T/2)|u - v|,
 \end{aligned}$$

where  $T$  is the length of the time interval of the problem. So, (4.2) holds by the Lipschitz constant

$$L_{CRE,1} := \frac{6 + T}{2}L_{b,1}.$$

### 4.2 The Lipschitz property of $\Phi_{b,CRE}$ for $p = 2$

We show that by the assumption (4.1) with Lipschitz constant  $L_{b,2}$  there exists  $L_{CRE,2} > 0$  constant s.t. for the increment function (3.5)

$$|\Phi_{b,CRE}(t, u, h) - \Phi_{b,CRE}(t, v, h)| \leq L_{CRE,2}|u - v|.$$

A straightforward calculation shows that

$$|\Phi_{b,CRE}(t, u, h) - \Phi_{b,CRE}(t, v, h)| \leq \frac{5 + h}{3}L_{b,2}|u - v| \leq \frac{5 + T}{3}L_{b,2}|u - v|,$$

therefore  $L_{b,CRE,2} = \frac{5+T}{3}L_{b,2}$  is a suitable Lipschitz constant.

### 4.3 Numerical experiment

We considered the scalar problem

$$\begin{cases} y' = -2t \sin y, & t \in [0, 1], \\ y(0) = 1. \end{cases} \tag{4.3}$$

The exact solution is  $y(t) = 2 \cot^{-1} \left( e^{t^2} \cot\left(\frac{1}{2}\right) \right)$ .

We solved the problem with the first order EE method (Table 1) and the second order MEE method (Table 2) as well as with their combinations with the CRE. For comparison we included the second-order Trapezoidal (TR) method and the third-order Heun's method (Table 3). We also compared our results with an implicit method, the implicit Trapezoidal (ITR) method, which is known to be a symmetric scheme [5]. The global errors were calculated in absolute value at the end of the time interval  $[0, 1]$  for time step sizes, successively



**Table 1.** Global errors at  $T = 1$  in absolute value in Example (4.3) obtained by the EE, Trapezoidal (TR) and EE + CRE methods and reducing  $h$  by a factor of 2.

<b>h</b>	<b>EE</b>	<b>Order</b>	<b>TR</b>	<b>Order</b>	<b>EE + CRE</b>	<b>Order</b>
0.1	1.9948e-02		1.0401e-03		7.8397e-04	
		1.0926		1.9514		2.1059
0.05	9.3539e-03		2.6893e-04		1.8212e-04	
		1.0449		1.9809		2.0511
0.025	4.5337e-03		6.8129e-05		4.3945e-05	
		1.0221		1.9915		2.0251
0.0125	2.2324e-03		1.7133e-05		1.0797e-05	

**Table 2.** Global errors at  $T = 1$  in absolute value in Example (4.3) obtained by the modified Euler (MEE), Heun's and MEE + CRE methods and reducing  $h$  by a factor of 2.

<b>h</b>	<b>MEE</b>	<b>Order</b>	<b>Heun's</b>	<b>Order</b>	<b>MEE + CRE</b>	<b>Order</b>
0.1	7.8397e-04		1.3543e-05		1.8774e-05	
		2.1059		3.2473		3.1410
0.05	1.8212e-04		1.4262e-06		2.1282e-06	
		2.0511		3.1289		3.0715
0.025	4.3945e-05		1.6304e-07		2.5317e-07	
		2.0251		3.0658		3.0360
0.0125	1.0797e-05		1.9472e-08		3.0867e-08	

**Table 3.** Global errors at  $T = 1$  in absolute value in Example (4.3) obtained by the implicit Trapezoidal (ITR) method as underlying method and reducing  $h$  by a factor of 2.

<b>h</b>	<b>ITR</b>	<b>Order</b>	<b>ITR + CRE</b>	<b>Order</b>
0.1	1.2317e-03		1.5204e-07	
		2.0011		3.7843
0.05	3.0770e-04		1.1035e-08	
		2.0003		3.8990
0.025	7.6911e-05		7.3968e-10	
		2.0001		3.9512
0.0125	1.9227e-05		4.7821e-11	

reduced by a factor of 2. The columns under ‘‘Order’’ show estimations of the convergence order by the formula

$$p \approx \frac{\ln(E(h)) - \ln(E(h/2))}{\ln 2},$$

where  $E(h)$  stands for the global error obtained with time-step size  $h$ .

As expected, the EE method behaves as a first order, while the MEE, TR,

and ITR as second order convergent methods. After the application of CRE, we get estimated orders around 2 for the EE method, 3 for the MEE and TR methods, and 4 for the ITR method. So the order of convergence was successfully increased by one with the method of classical Richardson extrapolation, and the order of convergence is increased by two when the BM is symmetric. Note that the EE + CRE method outperforms the TR method for all the studied time steps, but Heun's method is more accurate than MEE + CRE for each of the studied time step lengths. We remark however that as far as efficiency is considered, CRE allows parallelization, which can significantly reduce the computational time.

## 5 Conclusions

We gave a consistency analysis for any explicit one-step numerical method of order  $p = 1$  or  $p = 2$ , combined with the classical Richardson extrapolation when applied to a non-autonomous Cauchy problem. We showed that the CRE increases the order of consistency by one. Since the Lipschitz property of the time-stepping function of the method is inherited by the time-stepping function of the combined (BM + CRE) method, the same order of convergence is guaranteed if the base method possesses the Lipschitz property.

## Acknowledgements

The authors are thankful to the unknown referees for their constructive comments. "Application Domain Specific Highly Reliable IT Solutions" project has been implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the Thematic Excellence Programme TKP2020-NKA-06 (National Challenges Subprogramme) funding scheme. The project has been supported by the Hungarian Scientific Research Fund OTKA SNN125119 and K137699.

## References

- [1] C.E. Bréhier. Introduction to numerical methods for Ordinary Differential Equations. Lecture - Cours donné à Pristina (Kosovo), en décembre 2016, dans le cadre de l'école de recherche Franco-Kosovarde en Mathématiques, December 2016.
- [2] K. Burrage. *Parallel and Sequential Methods for Ordinary Differential Equations*. Oxford University Press, Oxford, New York, 1992.
- [3] J.C. Butcher. *Numerical Methods for Ordinary Differential Equations*. Second edition, Wiley, New York, 2003. <https://doi.org/10.1002/0470868279>.
- [4] I. Faragó, Á. Havasi and Z. Zlatev. The convergence of explicit Runge-Kutta methods combined with Richardson extrapolation. In J. Brandts, J. Chleboun, S. Korotov, K. Segeth, J. Sístek and T. Vejchodský (Eds.), *Applications of Mathematics 2012*, Navier-Stokes Equations and Related Nonlinear Problems, pp. 99–106, Prague, 2012. Institute of Mathematics AS CR. Available from Internet: <http://eudml.org/doc/287853>.

- [5] A. Gorgey. Theoretical analysis of symmetric Runge–Kutta methods. *Malaysian Journal of Mathematical Sciences*, **12**(3):369–382, 2018.
- [6] E. Hairer, C. Lubich and G. Wanner. *Geometric numerical integration: structure-preserving algorithms, 2nd edition*. Springer-Verlag, Berlin, 2006.
- [7] E. Hairer, S.P. Nørsett and G. Wanner. *Solving Ordinary Differential Equations I: Nonstiff Problems*. Springer-Verlag, Berlin, 1987.
- [8] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems*. Springer-Verlag, Berlin, 1991. <https://doi.org/10.1007/978-3-662-09947-6>.
- [9] W. Hundsdorfer and J.G. Verwer. *Numerical Method for Ordinary Differential Systems: The Initial Value Problem*. John Wiley & Sons Ltd., New York, 1991.
- [10] J.D. Lambert. *Numerical Methods for Ordinary Differential Equations*. Springer-Verlag, Berlin, 2003.
- [11] L.F. Richardson. The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, **210**(5):307–357, 1911. <https://doi.org/10.1098/rsta.1911.0009>.
- [12] L.F. Richardson. The deferred approach to the limit I. single lattice. *Philosophical Transactions of the Royal Society of London. Series A*, pp. 299–349, 1927.
- [13] E. Süli. *Numerical Solution of Ordinary Differential Equations*. University of Oxford, 2014.
- [14] Z. Zlatev, I. Dimov, I. Faragó, K. Georgiev, Á. Havasi and Tz. Ostromsky. Application of Richardson extrapolation for multi-dimensional advection equations, computers and mathematics with applications. *Computers and Mathematics with Applications*, **67**(12):2279–2293, 2014. <https://doi.org/10.1016/j.camwa.2014.02.028>.
- [15] Z. Zlatev, I. Dimov, I. Faragó and Á. Havasi. *Richardson Extrapolation - Practical Aspects and Applications*. De Gruyter, 2017.
- [16] Z. Zlatev, I. Faragó and Á. Havasi. Stability of the Richardson extrapolation together with the  $\theta$ -method. *Journal of Computational and Applied Mathematics*, **235**:507–517, 2010. <https://doi.org/10.1016/j.cam.2010.05.052>.