Mathematical Modelling and Analysis
Volume 24, Issue 1, 34–42, 2019
https://doi.org/10.3846/mma.2019.003

http://mma.vgtu.lt
ISSN: 1392-6292
eISSN: 1648-3510

On Discrete Value Distribution of Certain Compositions

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Received June 7, 2018; revised October 10, 2018; accepted October 12, 2018

Abstract. In the paper, we obtain universality theorems and a lower estimate for the number of zeros for the composition $F\left(\underline{\zeta}\left(s, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}\right)\right)$, where F is an operator in the space of analytic functions satisfying the Lipschitz type condition, and $\underline{\zeta}\left(s, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}\right)$ is a collection consisting of periodic and periodic Hurwitz zeta-functions.

Keywords: Mergelyan theorem, periodic Hurwitz zeta-function, periodic zeta-function, Rouché theorem, universality.

AMS Subject Classification: 11M41; 41A30.

1 Introduction

Let $s = \sigma + it$ be a complex variable, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$. The periodic zeta-function $\zeta(s; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}$$

and can be continued meromorphically to the whole complex plane with unique simple pole at the point $s = 1$ with residue $a = \frac{1}{k}\sum_{m=1}^{k} a_m$. If $a = 0$, then $\zeta(s;\mathfrak{a})$ is an entire function.

Let $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, be one more periodic sequence of complex numbers with minimal period $l \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$ with parameter α, $0 < \alpha \leqslant 1$, is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}$$

and can be continued meromorphically to the whole complex plane with unique simple pole at the point $s = 1$ with residue $b = \frac{1}{l}\sum_{m=0}^{l-1} b_m$. If $b = 0$, then $\zeta(s,\alpha;\mathfrak{b})$ is an entire function.

This note is devoted to discrete value distribution of collections consisting of periodic and periodic Hurwitz zeta-functions. In [2], the approximation of a collection of analytic functions by discrete shifts of the above collections of zeta-functions has been considered. For $j = 1,\dots,r_1$, let $\mathfrak{a}_j = \{a_{jm} : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q_j \in \mathbb{N}$, and $\zeta(s;\mathfrak{a}_j)$ be the corresponding periodic zeta-function. For $j = 1,\dots,r_2$, let $l_j \in \mathbb{N}$, $0 < \alpha_j \leqslant 1$, $\mathfrak{b}_{jl} = \{b_{jlm} : m \in \mathbb{N}_0\}$, $l = 1,\dots,l_j$, be a periodic sequence of complex numbers with minimal period q_{jl}, and let $\zeta(s,\alpha_j;\mathfrak{b}_{jl})$ be the corresponding periodic Hurwitz zeta-function. Moreover, let q denote the least common multiple of the periods $q_1,\dots,q_{r_1}$, and let $\eta_1,\dots,\eta_r$ be the reduced residue system modulo q, where $r = \varphi(q)$ is the Euler totient function. Similarly, let q_j denote the least common multiple of the periods $q_{1l_1},\dots,q_{jl_j}$, $j = 1,\dots,r_2$. Define the matrices

$$A = \begin{pmatrix} a_{1\eta_1} & a_{2\eta_1} & \dots & a_{r_1\eta_1} \\ a_{1\eta_2} & a_{2\eta_2} & \dots & a_{r_1\eta_2} \\ \dots & \dots & \dots & \dots \\ a_{1\eta_r} & a_{2\eta_r} & \dots & a_{r_1\eta_r} \end{pmatrix},$$

$$B_j = \begin{pmatrix} b_{j10} & b_{j20} & \dots & b_{jl_j0} \\ b_{j11} & b_{j21} & \dots & b_{jl_j1} \\ \dots & \dots & \dots & \dots \\ b_{j1(q_j-1)} & b_{j2(q_j-1)} & \dots & b_{jl_j(q_j-1)} \end{pmatrix}, \quad j = 1,\dots,r_2.$$

For the statement of a joint discrete universality theorem, we use the following notation. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$, $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, $H(K)$ with $K \in \mathcal{K}$ be the class of continuous functions on K that are analytic in the interior of K, and let $H_0(K)$ be the subclass of $H(K)$ of non-vanishing functions on K. Denote by $\mathbb{P}$ the set of all prime numbers, by $\# A$ the cardinality of the set A, and define the set

$$L(\mathbb{P};\alpha_1,\dots,\alpha_{r_2};h,\pi) = \left\{(\log p : p \in \mathbb{P}), (\log(m+\alpha_j) : m \in \mathbb{N}_0,\, j = 1,\dots,r_2), \frac{2\pi}{h}\right\}$$

with $h > 0$. Then the main result of [2] is the following theorem.

Theorem 1. *Suppose that the sequences* $\mathfrak{a}_1, \dots, \mathfrak{a}_{r_1}$ *are multiplicative,* $\operatorname{rank} A = r_1$*, the set* $L(\mathbb{P}; \alpha_1, \dots, \alpha_{r_2}; h, \pi)$ *is linearly independent over the field of rational numbers* $\mathbb{Q}$*, and* $\operatorname{rank} B_j = l_j$*,* $j = 1, \dots, r_2$*. Let* $K_j \in \mathcal{K}$*,* $j = 1, \dots, r_1$*,* $K_{jl} \in \mathcal{K}$*,* $j = 1, \dots, r_2$*,* $l = 1, \dots, l_j$*, and* $f_j(s) \in H_0(K_j)$*,* $j = 1, \dots, r_1$*,* $f_{jl}(s) \in H(K_{jl})$*,* $j = 1, \dots, r_2$*,* $l = 1, \dots, l_j$*. Then, for every* $\varepsilon > 0$*,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r_1} \sup_{s\in K_j} |\zeta(s+ikh; \mathfrak{a}_j) - f_j(s)| < \varepsilon,$$
$$\sup_{1\leqslant j\leqslant r_2} \sup_{1\leqslant j\leqslant l_j} \sup_{s\in K_{jl}} |\zeta(s+ikh, \alpha_j; \mathfrak{b}_{jl}) - f_{jl}(s)| < \varepsilon\Big\} > 0.$$

We note that N runs non-negative integers. Theorem 1 has the following modification.

Theorem 2. *Under hypotheses of Theorem 1, the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r_1} \sup_{s\in K_j} |\zeta(s+ikh; \mathfrak{a}_j) - f_j(s)| < \varepsilon,$$
$$\sup_{1\leqslant j\leqslant r_2} \sup_{1\leqslant j\leqslant l_j} \sup_{s\in K_{jl}} |\zeta(s+ikh, \alpha_j; \mathfrak{b}_{jl}) - f_{jl}(s)| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$*.*

We recall that $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. The aim of this paper is to obtain some analytic properties of the function $F\left(\underline{\zeta}\left(s, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}\right)\right)$ for a certain class of operators $F : H^\kappa(D) \to H(D)$, where

$$\underline{\zeta}\left(s, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}\right) = (\zeta(s; \mathfrak{a}_1), \dots, \zeta(s; \mathfrak{a}_{r_1}), \zeta(s, \alpha_1; \mathfrak{b}_{11}) \dots, \zeta(s, \alpha_1; \mathfrak{b}_{1l_1}), \dots,$$
$$\zeta(s, \alpha_{r_2}; \mathfrak{b}_{r_2 1}) \dots, \zeta(s, \alpha_{r_2}; \mathfrak{b}_{r_2 l_{r_2}}))$$

with $\underline{\alpha} = (\alpha_1, \dots, \alpha_{r_1})$, $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_{r_1})$, $\underline{\mathfrak{b}} = (\mathfrak{b}_{11}, \dots, \mathfrak{b}_{1l_1}, \dots, \mathfrak{b}_{r_2 1}, \dots, \mathfrak{b}_{r_2 l_{r_2}})$, and $\kappa = r_1 + \sum_{j=1}^{r_2} l_j$.

The space $H(D)$ is metrisable. There exists a sequence of compact sets $\{K_l : l \in \mathbb{N}\} \subset D$ such that $D = \bigcup_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact, then $K \subset K_l$ for some l. Then

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D),$$

is a metric on $H(D)$ inducing its topology of uniform convergence on compacta. Setting

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1\leqslant m\leqslant\kappa} \left(\rho(g_{1m}, g_{2m})\right), \quad \underline{g}_j = (g_{j1}, g_{j2}, \dots, g_{j\kappa}) \subset H^\kappa(D),\ j = 1, 2,$$

we obtain the metric which induces the product topology of $H^\kappa(D)$.

We note that the sets K_l can be chosen with connected complements. For example, we can take closed rectangles.

Suppose that $\beta_1, \dots, \beta_\kappa$ are positive numbers. We say that an operator $F : H^\kappa(D) \to H(D)$ belongs to the class $Lip(\beta_1, \dots, \beta_\kappa)$ if the following conditions hold:

1° For every polynomial $p = p(s)$ and sets $K_1, \dots, K_{r_1} \in \mathcal{K}$, there exists

$$\underline{g} = \left(g_1, \dots, g_{r_1}, g_{11}, \dots, g_{1l_1}, \dots, g_{r_2 1}, \dots, g_{r_2 l_{r_2}}\right) \in F^{-1}\{p\} \subset H^\kappa(D)$$

such that $g_j(s) \neq 0$ on K_j for $j = 1, \dots, r_1$;

2° For all $K \in \mathcal{K}$, there exist a constant $c > 0$ and sets $K_1, \dots, K_\kappa \in \mathcal{K}$ such that

$$\sup_{s \in K} |F(g_{11}(s), \dots, g_{1\kappa}(s)) - F(g_{21}(s), \dots, g_{2\kappa}(s))| \leqslant c \sup_{1 \leqslant j \leqslant \kappa} \sup_{s \in K_j} |g_{1j}(s) - g_{2j}(s)|^{\beta_j}$$

for all $(g_{j1}, \dots, g_{j\kappa}) \in H^\kappa(D)$, $j = 1, 2$.

We will prove the following discrete universality theorem on the approximation of analytic functions.

Theorem 3. *Suppose that $F \in Lip(\beta_1, \dots, \beta_\kappa)$, the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_{r_1}$ are multiplicative,* rank$A = r_1$, *the set $L(\mathbb{P}; \alpha_1, \dots, \alpha_{r_2}; h, \pi)$ is linearly independent over $\mathbb{Q}$, and* rank$B_j = l_j$, $j = 1, \dots, r_2$. *Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s \in K} \left| F\left(\underline{\zeta}\left(s, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}\right)\right) - f(s) \right| < \varepsilon \right\} > 0.$$

It is not difficult to give an example of $F \in Lip(\beta_1, \dots, \beta_\kappa)$. Actually, for a given $\left(g_1, \dots, g_{r_1}, g_{11}, \dots, g_{1l_1}, \dots, g_{r_2 1}, \dots, g_{r_2 l_{r_2}}\right) \in H^\kappa(D)$, we take

$$F\left(g_1, \dots, g_{r_1}, g_{11}, \dots, g_{1l_1}, \dots, g_{r_2 1}, \dots, g_{r_2 l_{r_2}}\right) = c_1 g_1^{(n_1)} + \dots + c_{r_1} g_{r_1}^{(n_{r_1})} + c_{11} g_{11}^{(n_{11})} + \dots + c_{1l_1} g_{1l_1}^{(n_{1l_1})} + \dots + c_{r_2 1} g_{r_2 1}^{(n_{r_2 1})} + \dots + c_{r_2 l_{r_2}} g_{r_2 l_{r_2}}^{(n_{r_2 l_{r_2}})},$$

where $c_1, \dots, c_{r_1}, c_{11}, \dots, c_{1l_1}, \dots, c_{r_2 1}, \dots, c_{r_2 l_{r_2}} \in \mathbb{C} \setminus \{0\}$ and $n_1, \dots, n_{r_1}, n_{11}, \dots, n_{1l_1}, \dots, n_{r_2 1}, \dots, n_{r_2 l_{r_2}} \in \mathbb{N}$. Then, for every polynomial $p = p(s)$, there exists $\underline{g} \in F^{-1}\{p\}$ such that $g_j(s) \neq 0$ on K_j, $j = 1, \dots, r_1$. Suppose that

$$p(s) = a_n s^n + \dots + a_0 \quad \text{with}\, a_n \neq 0.$$

Then we can take $\underline{g} = \left(a_1, \dots, a_{r_1}, b_{11}, \dots, b_{1l_1}, \dots, b_{r_2 1}, \dots, b_{r_2(l_{r_2}-1)}, g_{r_2 l_{r_2}}\right)$ with $a_1, \dots, a_{r_1} \in \mathbb{C} \setminus \{0\}$, $b_{11}, \dots, b_{1l_1}, \dots, b_{r_2 1}, \dots, b_{r_2(l_{r_2}-1)} \in \mathbb{C}$ and

$$g_{r_2 l_{r_2}}(s) = \frac{1}{c_{r_2 l_{r_2}}} \left(\frac{a_n s^{n + n_{r_2 l_{r_2}}}}{(n+1) \cdots (n + n_{r_2 l_{r_2}})} + \dots + \frac{a_0 s^{n_{r_2 l_{r_2}}}}{1 \cdots n_{r_2 l_{r_2}}} \right).$$

This shows that the condition 1° of the definition of the class $Lip(\beta_1, \dots, \beta_\kappa)$ is fulfilled.

For checking the condition 2° of the class $Lip(\beta_1, \dots, \beta_\kappa)$, we apply the Cauchy integral formula. Let $K \in \mathcal{K}$, and $K \subset G \subset \hat{K}$ with an open set G and $\hat{K} \in \mathcal{K}$. We take a closed simple contour L lying in $\hat{K} \setminus G$ and enclosing the set K. Taking $g_{j1}, \dots, g_{j\kappa} \in H^\kappa(D)$, $j = 1, 2$, and using the Cauchy integral formula, we find that, for $s \in K$,

$$\begin{aligned}
&|F(g_{11}(s), \dots, g_{1\kappa}(s)) - F(g_{21}(s), \dots, g_{2\kappa}(s))| \\
&\quad = \left| \sum_{m=1}^{\kappa} c_m \frac{n_m!}{2\pi i} \int_L \frac{g_{1m}(z) - g_{2m}(z)}{(z-s)^{n_m+1}} \, dz \right| \leqslant \sum_{m=1}^{\kappa} |c_m| |\hat{c}_m| \sup_{s \in L} |g_{1m}(s) - g_{2m}(s)| \\
&\quad \leqslant c \sup_{1 \leqslant m \leqslant \kappa} \sup_{s \in \hat{K}} |g_{1m}(s) - g_{2m}(s)|
\end{aligned} \tag{1.1}$$

with positive constants $\hat{c}_m$, $m = 1, \dots, \kappa$, and c. For simplicity, here we have used the notation $c_{jl} g_{jl}^{(n_{jl})} = c_{r_1+l_1+\dots+l_{j-1}+l} \, g_{r_1+l_1+\dots+l_{j-1}+l}^{(r_1+l_1+\dots+l_{j-1}+l)}$, $j = 1, \dots, r_2$, $l = 1, \dots, l_j$. Thus, by (1.1), we have that the condition 2° is satisfied with $\beta_1 = \dots = \beta_\kappa = 1$ and $K_1 = \dots = K_\kappa = \hat{K}$.

Theorem 3, as Theorem 1, has the following modification.

Theorem 4. *Under hypotheses of Theorem 3, the limit*

$$\lim_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s \in K} \left| F\left(\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}})\right) - f(s) \right| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Theorems 1 and 2 are called joint discrete universality theorem for zeta-functions with periodic coefficients. Theorems 3 and 4 are discrete universality theorems for composite functions of zeta-functions with periodic coefficients.

Theorem 3 contains a certain information on zeros of the function $F\left(\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}})\right)$.

Theorem 5. *Suppose that $F \in Lip(\beta_1, \dots, \beta_\kappa)$, the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_{r_1}$ are multiplicative,* $\operatorname{rank} A = r_1$, *the set $L(\mathbb{P}; \alpha_1, \dots, \alpha_{r_2}; h, \pi)$ is linearly independent over $\mathbb{Q}$, and* $\operatorname{rank} B_j = l_j$, *$j = 1, \dots, r_2$. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, F, \underline{\alpha}, \underline{\mathfrak{a}}, \underline{\mathfrak{b}}) > 0$ such that the function $F\left(\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}})\right)$, for sufficiently large N, has a zero in the disc*

$$|s - (\sigma_1 + \sigma_2)/2| \leqslant (\sigma_2 - \sigma_1)/2$$

for more than cN numbers k, $0 \leqslant k \leqslant N$.

2 Proof of universality theorems

We remind the Mergelyan theorem on the approximation of analytic functions by polynomials [3].

Lemma 1. *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and $f(s)$ be a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon/2. \tag{2.1}$$

Next proof of Theorem 3 follows.

Proof. Let $\beta = \min_{1\leqslant j\leqslant\kappa} \beta_j = \min\left(\min_{1\leqslant j\leqslant r_1} \beta_j, \min_{1\leqslant j\leqslant r_2} \min_{1\leqslant j\leqslant l_j} \beta_{jl}\right)$. In view of the condition 1° of the class $Lip(\beta_1,\dots,\beta_\kappa)$, for the polynomial $p=p(s)$ of Lemma 1 and every $K_1,\dots,K_{r_1}\in\mathcal{K}$, there exists an element

$$\underline{g} = \left(g_1,\dots,g_{r_1},g_{11},\dots,g_{1l_1},\dots,g_{r_2 1},\dots,g_{r_2 l_{r_2}}\right) \in F^{-1}\{p\}$$

such that $g_j(s)\neq 0$ on K_j for $j=1,\dots,r_1$. Suppose that $c>0$ is from condition 2° of the class $Lip(\beta_1,\dots,\beta_\kappa)$, $K_1,\dots,K_{r_1},K_{11},\dots,K_{1l_1},\dots,K_{r_2 1},\dots,$ $K_{r_2 l_{r_2}}$ correspond the set K in 2°, and that $k\in\mathbb{N}_0$ satisfies the inequalities

$$\sup_{1\leqslant j\leqslant r_1}\sup_{s\in K_j}|\zeta(s+ikh;\mathfrak{a}_j)-g_j(s)| < c^{-1/\beta}\left(\varepsilon/4\right)^{1/\beta}, \tag{2.2}$$

$$\sup_{1\leqslant j\leqslant r_2}\sup_{1\leqslant l\leqslant l_j}\sup_{s\in K_{jl}}|\zeta(s+ikh,\alpha_j;\mathfrak{b}_{jl})-g_{jl}(s)| < c^{-1/\beta}\left(\varepsilon/4\right)^{1/\beta}. \tag{2.3}$$

Then, for k satisfying the above inequalities, we find by 2° that

$$\begin{aligned}\sup_{s\in K}&\left|F\left(\underline{\zeta}\left(s+ikh,\underline{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}}\right)\right)-p(s)\right| = \sup_{s\in K}\left|F\left(\underline{\zeta}\left(s+ikh,\underline{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}}\right)\right)-F(\underline{g})\right|\\ &\leqslant c\sup_{1\leqslant j\leqslant r_1}\sup_{s\in K_j}|\zeta(s+ikh;\mathfrak{a}_j)-g_j(s)|^{\beta_j}\\ &+\sup_{1\leqslant j\leqslant r_2}\sup_{1\leqslant l\leqslant l_j}\sup_{s\in K_{jl}}|\zeta(s+ikh,\alpha_j;\mathfrak{b}_{jl})-g_{jl}(s)|^{\beta_{jl}} < 2cc^{-1}\frac{\varepsilon}{4}=\frac{\varepsilon}{2}.\end{aligned} \tag{2.4}$$

By Theorem 1, the set of $k\in\mathbb{N}_0$ satisfying inequalities (2.2) and (2.3) has a positive lower density. Therefore, in view of (2.4),

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}\left|F\left(\underline{\zeta}\left(s+ikh,\underline{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}}\right)\right)-p(s)\right|<\frac{\varepsilon}{2}\right\}>0. \tag{2.5}$$

Suppose that k satisfies the inequality

$$\sup_{s\in K}\left|F\left(\underline{\zeta}\left(s+ikh,\underline{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}}\right)\right)-p(s)\right|<\frac{\varepsilon}{2}.$$

Then, taking into account inequality (2.1), we have for such k

$$\begin{aligned}\sup_{s\in K}&\left|F\left(\underline{\zeta}\left(s+ikh,\underline{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}}\right)\right)-f(s)\right|\\ &\leqslant\sup_{s\in K}\left|F\left(\underline{\zeta}\left(s+ikh,\underline{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}}\right)\right)-p(s)\right|+\sup_{s\in K}|f(s)-p(s)|<\varepsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\{0\leqslant k\leqslant N:\sup_{s\in K}\left|F\left(\underline{\zeta}\left(s+ikh,\underline{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}}\right)\right)-p(s)\right|<\frac{\varepsilon}{2}\right\}\\ &\quad\subset\left\{0\leqslant k\leqslant N:\sup_{s\in K}\left|F\left(\underline{\zeta}\left(s+ikh,\underline{\alpha};\underline{\mathfrak{a}},\underline{\mathfrak{b}}\right)\right)-f(s)\right|<\varepsilon\right\},\end{aligned}$$

and the theorem follows by (2.5). □

Unfortunately, Theorem 4 does not follows directly from Theorem 2, therefore, we will give its direct proof.

Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X, let $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$, and, for $A \in \mathcal{B}(H(D))$, define

$$P_N(A) = \frac{1}{N+1} \#\left\{1 \leqslant k \leqslant N : \underline{\zeta}(s + ikh, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}) \in A\right\}.$$

Lemma 2. *Suppose that the sequences $\mathfrak{a}_1, \ldots, \mathfrak{a}_{r_1}$ are multiplicative,* $\operatorname{rank} A = r_1$*, the set $L(\mathbb{P}; \alpha_1, \ldots, \alpha_{r_2}; h, \pi)$ is linearly independent over $\mathbb{Q}$, and* $\operatorname{rank} B_j = l_j$*, $j = 1, \ldots, r_2$. Then P_N, as $N \to \infty$, converges weakly to a certain probability measure $P_{\underline{\zeta}}$ with support $S^{r_1} \times H(D)^{\kappa - r_1}$.*

The lemma is Proposition 3.1 of [2].

Lemma 3. *Suppose that $F \in Lip(\beta_1, \ldots, \beta_\kappa)$. Then*

$$P_{N,F}(A) \stackrel{def}{=} \frac{1}{N+1} \#\left\{0 \leqslant k \leqslant N : F\left(\underline{\zeta}\left(s+ikh, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}\right)\right) \in A\right\}, \; A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\underline{\zeta}} F^{-1}$ as $N \to \infty$. Moreover, the support of $P_{\underline{\zeta}} F^{-1}$ is the whole of $H(D)$.

Proof. We recall that $P_{\underline{\zeta}} F^{-1}(A) = P_{\underline{\zeta}}(F^{-1}A)$ for $A \in \mathcal{B}(H(D))$. The condition 2° of the class $Lip(\beta_1, \ldots, \beta_\kappa)$ shows that the operator F is continuous. Moreover, by the definitions of P_N and $P_{N,F}$, we have that $P_{N,F} = P_N F^{-1}$. Therefore, Lemma 2, Theorem 5.1 of [1] and the continuity of F prove the weak convergence of $P_{N,F}$ to $P_{\underline{\zeta}} F^{-1}$ as $N \to \infty$.

The condition 1° of the class $Lip(\beta_1, \ldots, \beta_\kappa)$ implies that, for each polynomial $p = p(s)$, there exists

$$\underline{g} = \left(g_1, \ldots, g_{r_1}, g_{11}, \ldots, g_{1l_1}, \ldots, g_{r_2 1}, \ldots, g_{r_2 l_{r_2}}\right) \\ \in \left(F^{-1}\{p\}\right) \cap \left(S^{r_1} \times H^{\kappa - r_1}(D)\right).$$

Actually, if $g_j(s) \neq 0$ on every $K_j \in \mathcal{K}$, $j = 1, \ldots, r_1$, then $g_j \in S$, $j = 1, \ldots, r_1$, because if $g_j(s) = 0$ on D, then in view of the equality $D = \bigcup_{l=1}^{\infty} \hat{K}_l$ with $\hat{K}_l \in \mathcal{K}$ from the definition of the metric ρ, we obtain $g_j(\hat{K}_l) = 0$ for some l.

We take an arbitrary $g \in H(D)$ and its open neighbourhood G. Then, by the continuity of F, the set $F^{-1}G$ is open as well. In virtue of Lemma 2, there exists a polynomial $p = p(s)$ lying in G. Therefore, $F^{-1}\{p\} \subset F^{-1}G$, and by the above remark, the set $F^{-1}G$ contains an element of $S^{r_1} \times H^{\kappa - r_1}(D)$. Hence, Lemma 2 implies the inequality $P_{\underline{\zeta}}(F^{-1}G) > 0$. Thus,

$$P_{\underline{\zeta}} F^{-1}(G) = P_{\underline{\zeta}}(F^{-1}G) > 0.$$

Since g and G are arbitrary, this shows that the support of the measure $P_{\underline{\zeta}} F^{-1}$ is the whole of $H(D)$. □

Next we give the proof of Theorem 4.

Proof. Let the polynomial $p(s)$ satisfy (2.1). Define the set

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \varepsilon/2 \right\}.$$

By the second part of Lemma 3, the set G_ε is an open neighbourhood of the element of the support of the measure $P_{\underline{\zeta}} F^{-1}$. Thus, $P_{\underline{\zeta}} F^{-1}(G_\varepsilon) > 0$. We recall that the set $A \in \mathcal{B}(H(D))$ is a continuity set of the measure $P_{\underline{\zeta}} F^{-1}$ if $P_{\underline{\zeta}} F^{-1}(\partial A) = 0$, where ∂A is a boundary of the set A. Define one more set

$$\hat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then the boundary $\partial \hat{G}_\varepsilon$ lies in the set

$$\left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\},$$

therefore, $\partial \hat{G}_{\varepsilon_1} \cap \partial \hat{G}_{\varepsilon_2} = \varnothing$ for distinct positive ε_1 and ε_2. This shows that the set $\hat{G}_\varepsilon$ is a continuity set of $P_{\underline{\zeta}} F^{-1}$ for all but at most countably many $\varepsilon > 0$. Moreover, the definitions of G_ε and $\hat{G}_\varepsilon$ together with (2.1) imply the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$. Hence,

$$P_{\underline{\zeta}} F^{-1}(\hat{G}_\varepsilon) \geqslant P_{\underline{\zeta}} F^{-1}(G_\varepsilon) > 0. \tag{2.6}$$

Using the equivalent of weak convergence of probability measures in terms of continuity sets, by the first part of Lemma 3 and (2.6), we obtain that

$$\lim_{N \to \infty} P_{N,F}(\hat{G}_\varepsilon) = P_{\underline{\zeta}} F^{-1}(\hat{G}_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. This and the definitions of $P_{N,F}$ and $\hat{G}_\varepsilon$ prove the theorem. □

3 Proof of Theorem 5

For convenience, we remind the Rouché theorem.

Lemma 4. *Suppose that G is a domain in $\mathbb{C}$, K is a compact subset of G, and $f(s)$ and $g(s)$ are analytic functions on G such that*

$$|f(s) - g(s)| < |f(s)|$$

for every point s in the boundary of K. Then $f(s)$ and $g(s)$ have the same number of zeros in the interior of K, taking into account multiplicities.

Proof of the lemma can be found, for example, in [4]. Now, we prove Theorem 5.

Proof. Let, for brevity,

$$\sigma_0 = \frac{\sigma_1 + \sigma_2}{2} \quad \text{and} \quad r_0 = \frac{\sigma_2 - \sigma_1}{2}.$$

We take $f(s) = s - \sigma_0$ in Theorem 3. Then, by the latter theorem, for every $\varepsilon > 0$, the set of $k \in \mathbb{N}_0$ satisfying the inequality

$$\sup_{|s-\sigma_0|\leqslant r_0} \left| F\left(\underline{\zeta}\left(s + ikh, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}\right)\right) - (s - \sigma_0) \right| < \varepsilon \tag{3.1}$$

has a positive lower density. We choose ε to satisfy

$$0 < \varepsilon < \frac{1}{20} \inf_{|s-\sigma_0|=r_0} |s - \sigma_0| = \frac{r_0}{20}.$$

Then we have that the functions $F\left(\underline{\zeta}\left(s + ikh, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}\right)\right)$ and $s - \sigma_0$ on the disc $|s - \sigma_0| \leqslant r_0$ satisfy the conditions of Lemma 4. Since, obviously, the function $s - \sigma_0$ has one zero in the disc $|s - \sigma_0| < r_0$, we find that also the function $F\left(\underline{\zeta}\left(s + ikh, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}\right)\right)$ has only one zero in that disc. However, the number of k satisfying inequality (3.1), for sufficiently large N, is greater than cN with a certain constant $c > 0$ depending on $\sigma_1, \sigma_2, F, \underline{\alpha}$, and $\underline{\mathfrak{a}}, \underline{\mathfrak{b}}$. The theorem is proved. □

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