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Global Existence Results for Functional Differential Inclusions with State-Dependent Delay

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Abstract. Our aim in this work is to study the existence of solutions of a functional differential inclusion with state-dependent delay. We use the Bohnenblust–Karlin fixed point theorem for the existence of solutions.

Keywords: functional differential inclusion, mild solution, infinite and state-dependent delay, fixed point, semigroup theory.

AMS Subject Classification: 34G20; 34K20; 34K30.

1 Introduction

In this work we shall prove the existence of solutions of a functional differential inclusion. Our investigations will be situated in the Banach space of real functions which are defined, continuous and bounded on the real axis $(-\infty,+\infty)$. We will use Bohnenblust–Karlin's fixed theorem, combined with the Corduneanu's compactness criteria. More precisely we will consider the following problem:

$$y'(t) - Ay(t) \in F(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0,+\infty), \tag{1.1}$$
$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{1.2}$$

where $F : J \times \mathcal{B} \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \in J$, and $(E, |\cdot|)$ is a real Banach space. $\mathcal{B}$ is the phase space to be specified later, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$. For any function y defined on $(-\infty, +\infty)$ and any $t \in J$ we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t. We assume that the histories y_t to some abstract phases $\mathcal{B}$, to be specified later.

For modeling scientific phenomena where the delay is either a fixed constant or is given as an integral in which case is called distributed delay, we use differential delay equations or functional differential equations ; see for instance the books [24, 30, 37].

An extensive theory is developed for evolution equations [2, 3, 21]. Uniqueness and existence results have been established recently for different evolution problems in the papers by Baghli and Benchohra for finite and infinite delay in [6, 7, 8, 9]. However, complicated situations in which the delay depends on the unknown functions have been proposed in modeling in recent years. These equations are frequently called equations with state-dependent delay. Over the past several years it has become apparent that equations with state-dependent delay arise also in several areas such as in classical electrodynamics [20], in population models [10], in models of commodity price fluctuations [11, 32], and in models of blood cell productions [33]. Existence results and among other things were derived recently for functional differential equations when the solution is depending on the delay on a bounded interval $[0, b]$ for impulsive problems. We refer the reader to the papers by Abada *et al.* [1], Ait Dads and Ezzinbi [16], Anguraj *et al.* [4], Hernandez *et al.* [27] and Li *et al.* [14, 31]. See also [5, 12, 25, 26, 35, 36].

To the best of our knowledge, there exist very few papers devoted to functional evolution inclusions with state-dependent delay on unbounded intervals. Those results are stated in the Fréchet space setting. So the present results initiate the study of such problems in the Banach space setting.

2 Preliminaries

In this section we present briefly some notations and definition, and theorem which are used throughout this work.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [23] and follow the terminology used in [28]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms:

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:

(i) $y_t \in \mathcal{B}$;

(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;

(iii) There exist two functions $L(\cdot), M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ independent of y with L continuous and bounded, and M bounded such that:

$$\|y_t\|_{\mathcal{B}} \le L(t) \sup\{|y(s)|: 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote

$$L = \sup\{L(t): t \in J\}, \quad M = \sup\{M(t): t \in J\}.$$

Remark 1. 1. (ii) is equivalent to $|\phi(0)| \le H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.

2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \le 0$.

3. From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0 :$ We necessarily have that $\phi(0) = \psi(0)$.

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E.

By $BC := BC(-\infty, +\infty)$ we denote the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in (-\infty, +\infty)} |y(t)|.$$

Finally, by $BC' := BC'([0, +\infty))$ we denote the Banach space of all bounded and continuous functions from $[0, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC'} = \sup_{t \in [0, +\infty)} |y(t)|.$$

Let (E, d) be a metric space. We use the following notations:

$$\mathcal{P}_{cl}(E) = \{Y \in \mathcal{P}(E): Y \text{ closed}\}, \qquad \mathcal{P}_{cv}(E) = \{Y \in \mathcal{P}(E): Y \text{ convex}\}.$$

Consider $H_d : \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A}, \mathcal{B}) = \max\Big\{ \sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d(\mathcal{A}, b)\Big\},$$

where $d(\mathcal{A}, b) = \inf_{a \in \mathcal{A}} d(a, b)$, $d(a, \mathcal{B}) = \inf_{b \in \mathcal{B}} d(a, b)$.

Definition 1. Let X, Y be Hausdorff topological spaces and $F : X \to \mathcal{P}(Y)$ is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $F(x)$ is a nonempty closed subset of X and if for each open set N of X containing $F(x)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$.

Let $(E, \|\cdot\|)$ be a Banach space. A multivalued map $A : E \to \mathcal{P}(E)$ has *convex (closed) values* if $A(x)$ is convex (closed) for all $x \in E$. We say that A is *bounded* on bounded sets if $A(B)$ is bounded in E for each bounded set B of E, i.e.,

$$\sup_{x\in B}\bigl\{\sup\bigl\{\|y\|\colon y \in A(x)\bigr\}\bigr\} < \infty.$$

F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(E)$. If the multivalued map F is completely continuous with non empty values, then F is u.s.c. if and only if F has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in F(x_n)$ implies $y_* \in F(x_*)$.

Definition 2. A function $F : J\times\mathcal{B} \longrightarrow \mathcal{P}(E)$ is said to be an L^1- Carathéodory multivalued map if it satisfies:

(i) $y \mapsto F(t, y)$ is upper semicontinuous for almost all $t \in J$;

(ii) $t \mapsto F(t, y)$ is measurable for each $y \in \mathcal{B}$;

(iii) for every positive constant l there exists $h_l \in L^1(J, \mathbb{R}^+)$

$$\bigl\|F(t, y)\bigr\| = \sup\bigl\{|v|\colon v \in F(t, y)\bigr\} \le h_l$$

for all $|y| \le l$ for almost all $t \in J$.

Lemma 1. *Let E be a Banach space. Let $F : J \times E \to \mathcal{P}_{cl,cv}(E)$ be a L^1- Carathéodory multivalued map; and let Γ be linear continuous from $L^1(J;E)$ into $C(J;E)$, then the operator*

$$\Gamma \circ S_F : C(J,E) \longrightarrow \mathcal{P}_{cp,cv}\bigl(C(J,X)\bigr),$$
$$y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(J;X) \times C(J;X)$.

Finally, we say that A has a *fixed point* if there exists $x \in E$ such that $x \in A(x)$.

For each $y : (-\infty, +\infty) \to E$ let the set $S_{F,y}$ known as the set of selectors from F defined by

$$S_{F,y} = \bigl\{v \in L^1(J;E)\colon v(t) \in F(t, y_{\rho(t,y_t)}),\ \text{a.e. } t \in J\bigr\}.$$

For more details on multivalued maps we refer to the books of Deimling [17], Denkowski *et al.* [18, 19], Górniewicz [22] and Hu and Papageorgiou [29].

Theorem 1 [Bohnenblust–Karlin fixed point [13]]. *Let $B \in \mathcal{P}_{cl,cv}(E)$. And $N : B \to \mathcal{P}_{cl,cv}(B)$ be an upper semicontinuous operator and $N(B)$ is a relatively compact subset of E. Then N has at least one fixed point in B.*

Let us assume that $\Omega \neq \emptyset$ is a subset of BC, and let $N : \Omega \to \Omega$ and consider the solutions of the equation

$$y(t) \in (Ny)(t). \tag{2.1}$$

Lemma 2 [Corduneanu [15]]. *Let $D \subset BC([0,+\infty), E)$. Then D is relatively compact if the following conditions hold:*

(a) *D is bounded in BC.*

(b) *The function belonging to D is almost equicontinuous on $[0,+\infty)$, i.e., equicontinuous on every compact subset of $[0,+\infty)$.*

(c) *The set $D(t) := \{y(t) \colon y \in D\}$ is relatively compact subset on every compact of $[0,+\infty)$.*

(d) *The function from D is equiconvergent, that is, given $\epsilon > 0$, responds $T(\epsilon) > 0$ such that $|u(t) - \lim_{t\to+\infty} u(t)| < \epsilon$, for any $t \geq T(\epsilon)$ and $u \in D$.*

3 Existence of Mild Solutions

Now we give our main existence result for problem (1.1)–(1.2). Before stating and proving this result, we give the definition of the mild solution.

Definition 3. We say that a continuous function $y : (-\infty,+\infty) \to E$ is a mild solution of problem (1.1)–(1.2) if $y(t) = \phi(t)$ for all $t \in (-\infty, 0]$, and the restriction of $y(\cdot)$ to the interval J is continuous and there exists $f(\cdot) \in L^1(J;E)$: $f(t) \in F(t, y_{\rho(t,y_t)})$ a.e. in J such that y satisfies the following integral equation

$$y(t) = T(t)\phi(t) - \int_0^t T(t-s)f(s)\,ds \quad \text{for each } t \in J. \tag{3.1}$$

Set

$$\mathcal{R}(\rho^-) = \big\{\rho(s,\phi) \colon (s,\phi) \in J \times \mathcal{B},\ \rho(s,\phi) \leq 0\big\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\| \leq \mathcal{L}^\phi(t)\|\phi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 2. The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [28].

Lemma 3. [27, Lemma 2.4] *If $y : (-\infty,+\infty) \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq \big(M + \mathcal{L}^\phi\big)\|\phi\|_{\mathcal{B}} + L\sup\big\{\big|y(\theta)\big|; \theta \in \big[0, \max\{0,s\}\big]\big\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $\mathcal{L}^\phi = \sup_{t\in\mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$.

Let us introduce the following hypotheses:

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$ which is compact for $t > 0$ in the Banach space E. Let $M' = \sup\{\|T\|_{B(E)} : t \geq 0\}$.

(H_2) The multifunction $F : J \times \mathcal{B} \longrightarrow \mathcal{P}(E)$ is Carathéodory with compact and convex values.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that:

$$H_d\big(F(t,u), F(t,v)\big) \leq k(t)\|u - v\|_{\mathcal{B}}$$

for each $t \in J$ and for all $u, v \in \mathcal{B}$ and

$$d\big(0, F(t,0)\big) \leq k(t)$$

with

$$k^* := \sup_{t\in J} \int_0^t k(s)\, ds < \infty. \tag{3.2}$$

Theorem 2. *Assume that* (H_1)–(H_3), (H_ϕ) *hold. If* $k^*M'L < 1$, *then the problem* (1.1)–(1.2) *has at least one mild solution on* BC.

Proof. Transform the problem (1.1)–(1.2) into a fixed point problem. Consider the multivalued operator $N : BC \to \mathcal{P}(BC)$ defined by:

$$N(y) := \left\{ h \in BC\colon h(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ T(t)\phi(0) + \int_0^t T(t-s) f(s)\, ds, & \text{if } t \in J, \end{cases} \right\}$$

where $f \in S_{F, y_{\rho(s,y_s)}}$.

Let $x(\cdot) : (-\infty, +\infty) \to E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0]; \\ T(t)\ \phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each $z \in BC$ with $z(0) = 0$, we denote by $\overline{z}$ the function

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0]; \\ z(t), & \text{if } t \in J, \end{cases}$$

if $y(\cdot)$ satisfies (3.1), we can decompose it as $y(t) = z(t) + x(t)$, $t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t T(t-s) f(s)\, ds, \quad t \in J,$$

where $f \in S_{F, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}}$. Set

$$BC_0' = \big\{ z \in BC' \colon z(0) = 0 \big\}$$

and let

$$\|z\|_{BC'_0} = \sup\{|z(t)| : t \in J\}, \quad z \in BC'_0.$$

BC'_0 is a Banach space with the norm $\|\cdot\|_{BC'_0}$.

We define the operator $\mathcal{A} : BC'_0 \to \mathcal{P}(BC'_0)$ by:

$$\mathcal{A}(z) := \left\{ h \in BC'_0 \colon h(t) = \begin{cases} 0, & \text{if } t \le 0; \\ \int_0^t T(t-s)f(s)\,ds, & \text{if } t \in J, \end{cases} \right\}$$

where $f \in S_{F, z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}$.

The operator A maps BC'_0 into BC'_0, indeed the map $\mathcal{A}(z)$ is continuous on $[0,+\infty)$ for any $z \in BC'_0$, $h \in \mathcal{A}(z)$ and for each $t \in J$ we have

$$\begin{aligned} |h(t)| &\le M' \int_0^t |f(s)|\,ds \\ &\le M' \int_0^t \left(k(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + |F(s,0)|\right) ds \\ &\le M' \int_0^t k(s)\,ds + M' \int_0^t k(s)\left(L|z(s)| + \left(M + \mathcal{L}^\phi + LM'H\right)\|\phi\|_{\mathcal{B}}\right) ds \\ &\le M'k^* + M' \int_0^t k(s)\left(L|z(s)| + \left(M + \mathcal{L}^\phi + LM'H\right)\|\phi\|_{\mathcal{B}}\right) ds. \end{aligned}$$

Set $C := \left(M + \mathcal{L}^\phi + LM'H\right)\|\phi\|_{\mathcal{B}}$. Then, we have

$$\begin{aligned} |h(t)| &\le M'k^* + M'C \int_0^t k(s)\,ds + M' \int_0^t L|z(s)|k(s)\,ds \\ &\le M'k^* + M'Ck^* + M'L\|z\|_{BC'_0} k^*. \end{aligned}$$

Hence, $\mathcal{A}(z) \in BC'_0$. Moreover, let $r > 0$ be such that

$$r \ge \frac{M'k^* + M'Ck^*}{1 - M'k^*L},$$

and B_r be the closed ball in BC'_0 centered at the origin and of radius r. Let $z \in B_r$ and $t \in [0,+\infty)$. Then

$$|h(t)| \le M'k^* + M'Ck^* + M'k^*Lr.$$

Thus $\|h\|_{BC'_0} \le r$, which means that the operator $\mathcal{A}$ transforms the ball B_r into itself. Now we prove that $\mathcal{A} : B_r \to \mathcal{P}(B_r)$ satisfies the assumptions of Bohnenblust–Karlin's fixed theorem. The proof will be given in several steps.

Step 1: We shall show that the operator $\mathcal{A}$ is closed and convex. This will be given in several claims.

Claim 1. $\mathcal{A}(z)$ is closed for each $z \in B_r$.

Let $(h_n)_{n\geq 0} \in \mathcal{A}(z)$ such that $h_n \to \tilde{h}$ in B_r. Then for $h_n \in B_r$ there exists $f_n \in S_{F,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}$ such that for each $t \in J$,

$$h_n(t) = \int_0^t T(t-s)f_n(s)\,ds.$$

Using the fact that F has compact values and from hypotheses (H_2), (H_3) we may pass a subsequence if necessary to get that f_n converges to $f \in L^1(J,E)$ and hence $f \in S_{F,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}$. Then for each $t \in J$,

$$h_n(t) \to \tilde{h}(t) = \int_0^t T(t-s)f(s)\,ds.$$

So, $\tilde{h} \in \mathcal{A}(z)$.

Claim 2. $\mathcal{A}(z)$ is convex for each $z \in B_r$.

Let $h_1, h_2 \in \mathcal{A}(z)$, then there exists $f_1, f_2 \in S_{F,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}}$ such that, for each $t \in J$ we have:

$$h_i(t) = \int_0^t T(t-s)f_i(s)\,ds, \quad i = 1,2.$$

Let $0 \leq \delta \leq 1$. Then, we have for each $t \in J$:

$$\big(\delta h_1 + (1-\delta)h_2\big)(t) = \int_0^t T(t-s)\big[\delta f_1(s) + (1-\delta)f_2(s)\big]\,ds.$$

Since F has convex values, one has $\delta h_1 + (1-\delta)h_2 \in \mathcal{A}(z)$.

Step 2: $\mathcal{A}(B_r) \subset B_r$ this is clear.

Step 3: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0,b]$ of $[0,+\infty)$ for $b > 0$. Let $\tau_1, \tau_2 \in [0,b], h \in \mathcal{A}(z)$ with $\tau_2 > \tau_1$, we have:

$$\begin{aligned}
\big|h(\tau_2) - h(\tau_1)\big| \quad &\leq \int_0^{\tau_1} \big\|T(\tau_2 - s) - T(\tau_1 - s)\big\|_{B(E)} \big|f(s)\big|\,ds \\
&+ \int_{\tau_1}^{\tau_2} \big\|T(\tau_2 - s)\big\|_{B(E)} \big|f(s)\big|\,ds \leq \int_0^{\tau_1} \big\|T(\tau_2 - s) - T(\tau_1 - s)\big\|_{B(E)} \\
&\times \big(k(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \big|F(s,0)\big|\big)\,ds \\
&+ \int_{\tau_1}^{\tau_2} \big\|T(\tau_2 - s)\big\|_{B(E)} \big(k(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \big|F(s,0)\big|\big)\,ds \\
\leq C &\int_0^{\tau_1} \big\|T(\tau_2 - s) - T(\tau_1 - s)\big\|_{B(E)} k(s)\,ds \\
&+ rL \int_0^{\tau_1} \big\|T(\tau_2 - s) - T(\tau_1 - s)\big\|_{B(E)} k(s)\,ds \\
&+ \int_0^{\tau_1} \big\|T(\tau_2 - s) - T(\tau_1 - s)\big\|_{B(E)} k(s)\,ds
\end{aligned}$$

$$+ C\int_{\tau_1}^{\tau_2}\big\|T(\tau_2-s)\big\|_{B(E)}k(s)\,ds + rL\int_{\tau_1}^{\tau_2}\big\|T(\tau_2-s)\big\|_{B(E)}k(s)\,ds$$
$$+\int_{\tau_1}^{\tau_2}\big\|T(\tau_2-s)\big\|_{B(E)}k(s)\,ds.$$

When $\tau_2 \to \tau_2$, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$, implies the continuity in the uniform operator topology (see [34]), this proves the equicontinuity.

Step 4: $\mathcal{A}(B_r)$ is relatively compact on every compact interval of $[0,\infty)$.

Let $t \in [0,b]$ for $b > 0$ and let ε be a real number satisfying $0 < \varepsilon < t$. For $z \in B_r$ we define

$$h_\varepsilon(t) = T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon)f(s)\,ds.$$

Note that the set

$$\left\{\int_0^{t-\varepsilon} T(t-s-\varepsilon)f(s)\,ds\colon z \in B_r\right\}$$

is bounded:

$$\left|\int_0^{t-\varepsilon} T(t-s-\varepsilon)f(s)\,ds\right| \le r.$$

Since $T(t)$ is a compact operator for $t > 0$, the set, $\big\{h_\varepsilon(t)\colon z \in B_r\big\}$ is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $z \in B_r$ we have

$$\begin{aligned}\big|h(t)-h_\varepsilon(t)\big| &\le M'\int_{t-\varepsilon}^{t}\big|f(s)\big|\,ds \le M'\int_{t-\varepsilon}^{t}k(s)\,ds\\ &\quad + M'C\int_{t-\varepsilon}^{t}k(s)\,ds + rM'\int_{t-\varepsilon}^{t}Lk(s)\,ds \ \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Therefore, the set $\{h(t)\colon z \in B_r\}$ is precompact, i.e., relatively compact.

Step 5: $\mathcal{A}$ has closed graph.

Let $\{z_n\}$ be a sequence such that $z_n \to z_*$, $h_n \in \mathcal{A}(z_n)$ and $h_n \to h_*$. We shall show that $h_* \in \mathcal{A}(z_*)$. Note, that $h_n \in \mathcal{A}(z_n)$ means that there exists $f_n \in S_{F,z^n_{\rho(s,z^n_s+x_s)}+x_{\rho(s,z^n_s+x_s)}}$ such that

$$h_n(t) = \int_0^t T(t-s)f_n(s)\,ds,$$

we must prove that there exists f_*

$$h_*(t) = \int_0^t T(t-s)f_*(s)\,ds.$$

Consider the linear and continuous operator $K : L^1(J,E) \to B_r$ defined by

$$K(v)(t) = \int_0^t T(t-s)v(s)\,ds,$$

we have

$$\big|K(f_n)(t) - K(f_*)(t)\big| = \big|h_n(t) - h_*(t)\big| \le \|h_n - h_*\|_\infty \to 0, \quad \text{as } n \to \infty.$$

From Lemma 2.2 it follows that $K \circ S_F$ is a closed graph operator and from the definition of K has $h_n(t) \in K \circ S_{F,z^n_{\rho(s,z^n_s+x_s)}+x_{\rho(s,z^n_s+x_s)}}$. As $z_n \to z_*$ and $h_n \to h_*$, there exists $f_* \in S_{F,z^*_{\rho(s,z^*_s+x_s)}+x_{\rho(s,z^*+x_s)}}$ such that:

$$h_*(t) = \int_0^t T(t-s)f_*(s)\,ds.$$

Hence the multivalued operator $\mathcal{A}$ is upper semi-continuous.

Step 6: $\mathcal{A}(B_r)$ is equiconvergent.

Let $z \in B_r$, we have, for $h \in \mathcal{A}(z)$:

$$\begin{aligned}
\big|h(t)\big| &\le M' \int_0^t \big|f(s)\big|\,ds \\
&\le M'k^* + M'C \int_0^t k(s)\,ds + M'r \int_0^t Lk(s)\,ds \\
&\le M'k^* + M'C \int_0^t k(s)\,ds + M'rL \int_0^t k(s)\,ds.
\end{aligned}$$

Then by (4), we have

$$\big|h(t)\big| \to l \le M'k^*(1 + C + rL), \quad \text{as } t \to +\infty.$$

Hence,

$$\big|h(t) - h(+\infty)\big| \to 0, \quad \text{as } t \to +\infty.$$

As a consequence of Steps 1–6, with Lemma 2, we can conclude that $\mathcal{A} : B_r \to \mathcal{P}(B_r)$ is continuous and compact. From Bohnenblust–Karlin's fixed theorem, we deduce that $\mathcal{A}$ has a fixed point z^*. Then $y^* = z^* + x$ is a fixed point of the operator N, which is a mild solution of the problem (1.1)–(1.2). $\square$

4 An Example

Consider the following functional partial differential equation

$$\frac{\partial}{\partial t} z(t,x) - \frac{\partial^2}{\partial x^2} z(t,x) \in F\big(t, z(t - \sigma\big(t, z(t,0)\big), x)\big), \tag{4.1}$$

$x \in [0, \pi]$, $t \in [0, +\infty)$,

$$z(t, 0) = z(t, \pi) = 0, \quad t \in [0, +\infty), \tag{4.2}$$

$$z(\theta, x) = z_0(\theta, x), \quad t \in (-\infty, 0], \ x \in [0, \pi], \tag{4.3}$$

where F is a given multivalued map, and $\sigma : (-\infty, +\infty) \to \mathbb{R}^+$ is continuous.

Take $E = L^2[0, \pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E, \ \omega, \omega' \text{ are absolutely continuous}, \ \omega'' \in E, \ \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2 (\omega, \omega_n)\omega_n, \quad \omega \in D(A),$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}} \sin ns$, $n = 1, 2, \ldots$ is the orthogonal set of eigenvectors of A. It is well known (see [34]) that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega, \omega_n)\omega_n, \quad \omega \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathcal{R}^-; E)$ and $\phi \in \mathcal{B}$, then (H_ϕ), where $\rho(t, \varphi) = t - \sigma(\varphi)$.

Then the problem (1.1)–(1.2) in an abstract formulation of the problem (4.1)–(4.3), and if the conditions (H_1)–(H_3), (H_ϕ) are satisfied, Theorem 2 implies that the problem (4.1)–(4.3) has at least one mild solutions on BC.

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