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General Control Modulo and Tauberian Remainder Theorems for $(C,1)$ Summability*

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Abstract. We prove for the $(C,1)$ summability method several Tauberian remainder theorems using the general control modulo of the oscillatory behavior.

Keywords: λ-bounded series, summability by $(C,1)$ method, general control modulo of the oscillatory behavior, Tauberian remainder theorems.

AMS Subject Classification: 40C99.

1 Introduction

A sequence $x=\{\xi_n\}$ is called bounded with the rapidity $\lambda=\{\lambda_n\}$ $(0<\lambda_n\uparrow\infty)$ if $\lambda_n(\xi_n-\xi)=O(1)$ with $\lim\xi_n=\xi$. Let

$$m^\lambda=\bigl\{x \bigm| x=\{\xi_n\}\wedge\lim\xi_n=\xi\wedge\lambda_n(\xi_n-\xi)=O(1)\bigr\}.$$

A sequence $x=\{\xi_n\}$ is called λ-bounded by Cesàro method $(C,1)$ if $(C,1)x$ is λ-bounded. That means

$$\lambda_n\bigl(\sigma_n(x)-\sigma(x)\bigr)=O(1) \tag{1.1}$$

with

$$\sigma_n(x)=\frac{1}{n+1}\sum_{k=0}^{n}\xi_k\wedge\lim_{n\to\infty}\sigma_n(x)=\sigma(x).$$

Shortly we note this fact

$$x\in\bigl((C,1),m^\lambda\bigr). \tag{1.2}$$

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G. Kangro [7] and I. Tammeraid [9] proved several Tauberian remainder theorems for method $(C,1)$ using summability with given rapidity. For example see one of them in [9].

Proposition 1. *If the sequences x and λ satisfy the conditions* (1.2)*,*

$$n\lambda_n \Delta\xi_n = O(1), \tag{1.3}$$

where

$$\Delta\xi_n = \begin{cases} \xi_n - \xi_{n-1}, & n \in \mathbf{N}, \\ \xi_0, & n = 0, \end{cases}$$

and

$$\frac{\lambda_n}{n+1} \sum_{k=0}^{n} \frac{1}{\lambda_k} = O(1), \tag{1.4}$$

then $x \in m^{\lambda}$.

If we study the condition (1.4) in the case $\lambda_n = (n+1)^{\alpha}$, we get that α has to satisfy the condition $0 < \alpha < 1$. That means we are not able to use the Tauberian remainder theorems proved by G. Kangro and I. Tammeraid in the case $\lambda_n = (n+1)^{\alpha}$ with $\alpha \geq 1$. Therefore we are interested in the presentations of G. H. Hardy [6], E. Landau [8], M. Dik [5], İ. Çanak, Ü. Totur and M. Dik [1, 2, 3, 4, 10, 11]. Firstly this concept was used in the paper [5]. So we prove several new Tauberian remainder theorems for $(C,1)$.

2 Tauberian Remainder Theorems for $(C,1)$

Let $\mathbf{N}$ be the set of all natural numbers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. The classical control modulo of the oscillatory behavior of the sequence $\{\xi_n\}$ is denoted by

$$\omega_n^{(0)}(x) = n\Delta\xi_n. \tag{2.1}$$

The general control modulo of the oscillatory behavior of order $m \in \mathbf{N}$ of sequence x is defined by

$$\omega_n^{(m)}(x) = \omega_n^{(m-1)}(x) - \sigma_n\big(\omega^{(m-1)}(x)\big) \quad (m \in \mathbf{N} \text{ and } n \in \mathbf{N}_0). \tag{2.2}$$

G.H. Hardy [6] proved that the condition $\omega_n^{(0)}(x) = O(1)$ is a Tauberian condition for the $(C,1)$ summability method.

E. Landau [8] extended Hardy's Tauberian condition as follows

$$\omega_n^{(0)}(x) \geq -M \quad (n \in \mathbf{N}_0) \tag{2.3}$$

for some $M > 0$.

M. Dik [5] showed that the condition (2.3) in E. Landau's statement can be replaced by the condition

$$\omega_n^{(1)}(x) \geq -M \quad (n \in \mathbf{N}_0).$$

İ. Çanak and Ü. Totur (see [1] and [3]) proved: if for some $\{M_n\}$ with $M_n > 0$

$$\omega_n^{(2)}(x) \geq -M_n \quad (n \in \mathbf{N}_0)$$

and x is $(C, 1)$ summable, then x is convergent.

Let us start with the simplest Tauberian theorem using the general control modulo.

Theorem 1. *If the condition* (1.2) *is satisfied,*

$$\lambda_n V_n^0(\Delta x) = O(1), \tag{2.4}$$

where $V_n^0(\Delta x) = \frac{1}{n+1}\sum_{k=0}^{n} \omega_k^{(0)}(x)$, *then* $x \in m^\lambda$.

Proof. As (see [1])

$$\xi_n - \sigma_n(x) = V_n^0(\Delta x),$$

then

$$\lambda_n\big(\xi_n - \sigma(x)\big) = \lambda_n\big(\sigma_n(x) - \sigma(x)\big) + \lambda_n V_n^0(\Delta x).$$

Using (1.2) and (2.4) we get the assertion of Theorem 1 is valid. □

Let us prove the following statement.

Lemma 1. *The assertion*

$$\omega_n^{(1)}(x) = \omega_n^{(0)}(x) - \xi_n + \sigma_n(x) \quad (n \in \mathbf{N}) \tag{2.5}$$

is valid.

Proof. Using (2.2) for $m = 1$ and (2.1) we get for $n \in \mathbf{N}$

$$\begin{aligned}
\omega_n^{(1)}(x) &= \omega_n^{(0)}(x) - \sigma_n\big(\omega^{(0)}(x)\big) \\
&= \omega_n^{(0)}(x) - \frac{1}{n+1}\sum_{k=0}^{n}\omega_k^{(0)}(x) = \omega_n^{(0)}(x) - \frac{1}{n+1}\sum_{k=0}^{n} k\Delta\xi_k \\
&= \omega_n^{(0)}(x) - \frac{1}{n+1}\left(\sum_{k=1}^{n} k\xi_k - \sum_{k=0}^{n}(k+1)\xi_k + (n+1)\xi_n\right) \\
&= \omega_n^{(0)}(x) - \xi_n + \frac{1}{n+1}\sum_{k=1}^{n}\xi_k = \omega_n^{(0)}(x) - \xi_n + \sigma_n(x).
\end{aligned}$$

That means the statement of Lemma 1 is valid. □

We use the result of Lemma 1 for the proof of the next theorem.

Theorem 2. *If the conditions*

$$\lambda_n \omega_n^{(0)}(x) = O(1), \tag{2.6}$$

$$\lambda_n \omega_n^{(1)}(x) = O(1) \tag{2.7}$$

and (1.2) *are satisfied, then* $x \in m^\lambda$.

Proof. Using (2.5) we get

$$\lambda_n\big(\xi_n - \sigma(x)\big) = \lambda_n \omega_n^{(0)}(x) - \lambda_n \omega_n^{(1)}(x) + \lambda_n\big(\sigma_n(x) - \sigma(x)\big).$$

As the conditions (1.2), (2.6) and (2.7) are satisfied the assertion of Theorem 2 is valid. □

Remark 1. The assertion (1.3) ⇔ (2.6) is valid.

Now we can prove the following result.

Lemma 2. *The assertion*

$$\omega_n^{(2)}(x) = \omega_n^{(0)}(x) - 2\xi_n + 3\sigma_n(x) - \frac{1}{n+1}\sum_{k=0}^{n}\sigma_k(x) \quad (n \in \mathbf{N})$$

is valid.

Proof. Let $\xi_{-1} = 0$. Using (2.2) for $m = 2$ and (2.5) we get

$$\begin{aligned}\omega_n^{(2)}(x) &= \omega_n^{(1)}(x) - \sigma_n\big(\omega^{(1)}(x)\big) \\ &= \omega_n^{(0)}(x) - \xi_n + \sigma_n(x) - \frac{1}{n+1}\sum_{k=0}^{n}\big(k\Delta\xi_k - \xi_k + \sigma_k(x)\big).\end{aligned}$$

As

$$\begin{aligned}\sum_{k=0}^{n}\big(k\Delta\xi_k - \xi_k + \sigma_k(x)\big) &= \sum_{k=0}^{n}\big(k(\xi_k - \xi_{k-1}) - \xi_k + \sigma_k(x)\big) \\ &= \sum_{k=0}^{n} k\xi_k - \sum_{k=0}^{n-1}(k+1)\xi_k - \sum_{k=0}^{n}\xi_k + \sum_{k=0}^{n}\sigma_k(x) \\ &= \sum_{k=0}^{n} k\xi_k - \sum_{k=0}^{n}(k+1)\xi_k + (n+1)\xi_n - \sum_{k=0}^{n}\xi_k + \sum_{k=0}^{n}\sigma_k(x) \\ &= -2\sum_{k=0}^{n}\xi_k + (n+1)\xi_n + \sum_{k=0}^{n}\sigma_k(x),\end{aligned}$$

then

$$\omega_n^{(2)}(x) = \omega_n^{(0)}(x) - 2\xi_n + 3\sigma_n(x) - \frac{1}{n+1}\sum_{k=0}^{n}\sigma_k(x) \quad (n \in \mathbf{N}).$$

That means the assertion of Lemma 2 is valid. □

Using Lemma 2 we get the next assertion.

Theorem 3. *If the conditions* (1.2), (2.6),

$$\lambda_n \omega_n^{(2)}(x) = O(1), \tag{2.8}$$

$$\lambda_n\left(\frac{1}{n+1}\sum_{k=0}^{n}\sigma_k(x) - \sigma(x)\right) = O(1) \tag{2.9}$$

are satisfied, then $x \in m^{\lambda}$.

Proof. Using Lemma 2 we get

$$2\big(\xi_n - \sigma(x)\big) = \omega_n^{(0)}(x) - \omega_n^{(2)}(x) + 3\big(\sigma_n(x) - \sigma(x)\big) - \Bigg(\frac{1}{n+1}\sum_{k=0}^{n}\sigma_k(x) - \sigma(x)\Bigg)$$

and

$$2\lambda_n\big(\xi_n - \sigma(x)\big) = \lambda_n\omega_n^{(0)}(x) - \lambda_n\omega_n^{(2)}(x) + 3\lambda_n\big(\sigma_n(x) - \sigma(x)\big) - \lambda_n\Bigg(\frac{1}{n+1}\sum_{k=0}^{n}\sigma_k(x) - \sigma(x)\Bigg).$$

While the conditions (2.6), (2.8), (1.2) and (2.9) are satisfied we get

$$\lambda_n\big(\xi_n - \sigma(x)\big) = O(1) + O(1) + O(1) + O(1) = O(1).$$

That means the assertion of Theorem 3 is valid. □

Remark 2. The assertion (1.2) $\wedge$ (1.4) $\Rightarrow$ (2.9) is valid.

Analogically it is possible to prove Lemma 3 and Theorem 4.

Lemma 3. *The assertion*

$$\omega_n^{(3)}(x) = \omega_n^{(0)}(x) - 3\xi_n + 6\sigma_n(x) - \frac{4}{n+1}\sum_{k=0}^{n}\sigma_k(x) + \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+1}\sum_{\nu=0}^{k}\sigma_\nu(x) \quad (n \in \mathbf{N})$$

is valid.

Using Lemma 3 we get the following assertion.

Theorem 4. *If the conditions* (1.2), (2.6), (2.9),

$$\lambda_n\omega_n^{(3)}(x) = O(1),$$

and

$$\lambda_n\Bigg(\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+1}\sum_{\nu=0}^{k}\sigma_\nu(x) - \sigma(x)\Bigg) = O(1) \qquad (2.10)$$

are satisfied, then $x \in m^\lambda$.

Remark 3. The assertion (1.2) $\wedge$ (1.4) $\Rightarrow$ (2.10) is valid.

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