Mathematical Modelling and Analysis
Volume 14 Number 1, 2009, pages 79–89
Doi:10.3846/1392-6292.2009.14.79-89

ISSN 1392-6292 print
ISSN 1648-3510 online
www.vgtu.lt/mma/

Numerical Solution of Volterra Integral Equations with Weakly Singular Kernels which May Have a Boundary Singularity*

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Received October 30, 2008; published online February 25, 2009

Abstract. We propose a piecewise polynomial collocation method for solving linear Volterra integral equations of the second kind with kernels which, in addition to a weak diagonal singularity, may have a weak boundary singularity. Global convergence estimates are derived and a collection of numerical results is given.

Key words: Volterra integral equation, weakly singular kernel, boundary singularity, collocation method.

1 Introduction

Let $C^k(\Omega)$ be the set of all k times continuously differentiable functions on Ω, $C^0(\Omega) = C(\Omega)$. Let $b \in \mathbb{R} = (-\infty, \infty)$, $b > 0$,

$$D_b = \{(x, y) : 0 \le x \le b,\ 0 < y < x\}, \quad \overline{D}_b = \{(x, y) : 0 \le y \le x \le b\}.$$

In many practical applications (see, for example, [3, 5]) there arise integral equations of the form

$$u(x) = \int_0^x K(x, y)u(y)dy + f(x), \quad 0 \le x \le b, \tag{1.1}$$

with $f \in C^m[0, b]$, $K(x, y) = g(x, y)(x - y)^{-\nu}$, $0 < \nu < 1$, $g \in C^m(\overline{D}_b)$, $m \in \mathbb{N} = \{1, 2, \ldots\}$. The solution $u(x)$ to (1.1) is typically non-smooth at $x = 0$ where its derivatives become unbounded (see, for example, [3, 4, 5, 9]). In collocation methods the singular behaviour of the solution $u(x)$ can be taken into account by using polynomial splines on special graded grids

$$\Delta_N^r = \{x_0, \ldots, x_N : 0 = x_0 < \ldots < x_N = b\}$$

* This work was supported by EstSF grant No. 7353

with the nodes

$$x_i = b(i/N)^r, \quad i = 0, \ldots, N, \quad N \in \mathbb{N}, \quad r \in \mathbb{R}, \quad r \geq 1. \tag{1.2}$$

The parameter r characterizes the degree of non-uniformity of the grid Δ_N^r: if $r > 1$, then the nodes $x_0, \ldots, x_N$ of the grid Δ_N^r are more densely clustered near the left endpoint of the interval $[0, b]$ where $u(x)$ may be singular. By using a collocation method based on the grid Δ_N^r and piecewise polynomials of degree at most $m-1$ one can reach a convergence of order $\mathcal{O}(N^{-m})$ for $r \geq m/(1-\nu)$, see [3, 4, 5]. However, although the piecewise polynomial collocation method on Δ_N^r turns out to be stable for solving weakly singular integral equations (see [8]), the realization of this method in case of strongly graded grids Δ_N^r by large values of r may lead to unstable behaviour of numerical results.

To avoid problems associated with the use of strongly graded grids the following approach for solving (1.1) can be used: first we perform in (1.1) a change of variables so that the singularities of the derivatives of the solution will be milder or disappear and after that we solve the transformed equation by a collocation method on a mildly graded or uniform grid. We refer to [13] for details (see also [2, 7, 12]). Note that in [10, 15] similar ideas for solving Fredholm integral equations have been used (see also [6, 11, 16]).

In the present paper we extend the domain of applicability of this approach. To this aim, we examine a more complicated situation for equation (1.1) where the kernel $K(x, y)$, in addition to a diagonal singularity (a singularity as $y \to x$), may have a boundary singularity (a singularity as $y \to 0$). Actually, we assume that the kernel $K(x, y)$ has the form

$$K(x, y) = g(x, y)(x - y)^{-\nu} y^{-\lambda}, \quad (x, y) \in D_b, \quad 0 < \nu < 1, \quad 0 \leq \lambda < 1, \tag{1.3}$$

where $g \in C^m(\overline{D}_b)$, $m \in \{0\} \cup \mathbb{N}$. The set of kernels satisfying (1.3) will be denoted by $W^{m,\nu,\lambda}(D_b)$.

Throughout the paper c denotes a positive constant which may have different values by different occurrences.

2 Regularity of the Solution

For given $m \in \mathbb{N}$ and $0 < \theta < 1$ let $C^{m,\theta}(0, b]$ be the set of functions $u \in C[0, b] \cap C^m(0, b]$ such that

$$|u^{(j)}(x)| \leq c x^{1-\theta-j}, \quad 0 < x \leq b, \quad j = 1, \ldots, m. \tag{2.1}$$

It follows from [14] that the regularity of the solution to (1.1) can be characterized by the following result.

Lemma 1. *Assume that $K \in W^{m,\nu,\lambda}(D_b)$ and $f \in C^{m,\nu+\lambda}(0, b]$ where $m \in \mathbb{N}$, $0 < \nu < 1$, $0 \leq \lambda < 1$, $\nu + \lambda < 1$. Then equation* (1.1) *has a unique solution $u \in C^{m,\nu+\lambda}(0, b]$.*

3 Smoothing Transformation

For given $\varrho \in [1,\infty)$ denote

$$\varphi(s) = b^{1-\varrho} s^{\varrho}, \quad 0 \le s \le b. \tag{3.1}$$

Clearly, $\varphi \in C[0,b]$, $\varphi(0) = 0$, $\varphi(b) = b$ and $\varphi'(s) > 0$ for $0 < s \le b$. Thus, φ maps $[0,b]$ onto $[0,b]$ and has a continuous inverse $\varphi^{-1} : [0,b] \to [0,b]$,

$$\varphi^{-1}(x) = b^{(\varrho-1)/\varrho} x^{1/\varrho}, \quad 0 \le x \le b.$$

Note that $\varphi(s) \equiv s$ for $\varrho = 1$. We are interested in a transformation (3.1) with $\varrho > 1$ since it possesses a smoothing property for $u(\varphi(s))$ with singularities of $u'(x), \dots, u^{(m)}(x)$ at $x = 0$ (see Lemma 2).

Lemma 2. *Let $u \in C^{m,\theta}(0,b]$, $m \in \mathbb{N}$, $0 < \theta < 1$, and let φ be the transformation* (3.1)*. Furthermore, let*

$$u_\varphi(s) = u(\varphi(s)), \quad 0 \le s \le b.$$

Then $u_\varphi \in C[0,b] \cap C^m(0,b]$ and

$$|u_\varphi^{(j)}(s)| \le c s^{\varrho(1-\theta)-j}, \quad 0 < s \le b, \quad j = 1, \dots, m. \tag{3.2}$$

Proof. The smoothness claim is clear. Further, for the derivatives of the composite function $u_\varphi = u \circ \varphi$, we have the Faà di Bruno's representation

$$u_\varphi^{(j)}(s) = \sum \frac{j!}{n_1! \dots n_j!} u^{(n)}(\varphi(s)) \left(\frac{\varphi'(s)}{1!}\right)^{n_1} \cdots \left(\frac{\varphi^{(j)}(s)}{j!}\right)^{n_j}, \tag{3.3}$$

where $0 < s \le b$, $n = n_1 + \dots + n_j$ and the sum is taken over all $n_1, \dots, n_j \in \{0\} \cup \mathbb{N}$ for which $n_1 + 2n_2 + \dots + jn_j = j$, $j = 1, \dots, m$. It follows from (2.1), (3.1), $n = n_1 + \dots + n_j$ and $n_1 + 2n_2 + \dots + jn_j = j$ that

$$\left| u^{(n)}(\varphi(s))(\varphi'(s))^{n_1} \dots (\varphi^{(j)}(s))^{n_j} \right| \le c s^{\varrho(1-\theta)-j}, \quad 0 < s \le b.$$

This together with (3.3) yields (3.2). □

Remark 1. Instead of (3.1) other transformations are possible. We refer to [13] for a general discussion in this connection.

4 Numerical Method

Using (3.1) we introduce in (1.1) the change of variables $y = \varphi(s)$, $x = \varphi(t)$, $s, t \in [0,b]$. We obtain an integral equation of the form

$$u_\varphi(t) = \int_0^t K_\varphi(t,s) u_\varphi(s) ds + f_\varphi(t), \quad 0 \le t \le b, \tag{4.1}$$

where

$$f_\varphi(t) = f(\varphi(t)), \quad K_\varphi(t,s) = K(\varphi(t), \varphi(s))\varphi'(s)$$

are given functions and $u_\varphi(t) = u(\varphi(t))$ is a function which we have to find.

For given integers $m, N \in \mathbb{N}$ let

$$S^{(-1)}_{m-1}(\Delta^r_N) = \{v_N : v_N\big|_{[x_{j-1},x_j]} \in \pi_{m-1}, j = 1, \ldots, N\},$$
$$S^{(0)}_{m-1}(\Delta^r_N) = \{v_N \in C[0,b] : v_N\big|_{[x_{j-1},x_j]} \in \pi_{m-1}, j = 1, \ldots, N\}$$

be the underlying spline spaces of piecewise polynomial functions on the grid Δ^r_N with the nodes (1.2). Here $v_N\big|_{[x_{j-1},x_j]}$ $(j = 1, \ldots, N)$ is the restriction of $v_N(t)$, $t \in [0,b]$, to the subinterval $[x_{j-1}, x_j] \subset [0,b]$ and π_{m-1} denotes the set of polynomials of degree not exceeding $m-1$. Note that the elements of $S^{(-1)}_{m-1}(\Delta^r_N)$ may have jump discontinuities at the interior knots $x_1, \ldots, x_{N-1}$ of the grid Δ^r_N. In every subinterval $[x_{j-1}, x_j]$ $(j = 1, \ldots, N)$ we introduce $m \in \mathbb{N}$ interpolation (collocation) points

$$x_{jl} = x_{j-1} + \eta_l(x_j - x_{j-1})\,, \quad l = 1, \ldots, m;\ j = 1, \ldots, N\,, \tag{4.2}$$

where $\eta_1, \ldots, \eta_m$ are some fixed (collocation) parameters such that

$$0 \le \eta_1 < \ldots < \eta_m \le 1. \tag{4.3}$$

We find an approximation $v_N = v_{N,m,r,\varphi}$ to u_φ, the solution of equation (4.1) (under the conditions of Theorem 1 below the equations (1.1) and (4.1) are uniquely solvable), by collocation method from the following conditions:

$$v_N \in S^{(-1)}_{m-1}(\Delta^r_N)\,, \quad N, m \in \mathbb{N},\ r \ge 1\,, \tag{4.4}$$

$$v_N(x_{jl}) = \int_0^{x_{jl}} K_\varphi(x_{jl}, s) v_N(s)\, ds + f_\varphi(x_{jl}),\ \ l = 1, \ldots, m;\ j = 1, \ldots, N, \tag{4.5}$$

with x_{jl}, $l = 1, \ldots, m$; $j = 1, \ldots, N$, given by formula (4.2).

Having determined the approximation v_N for u_φ, we determine an approximation $u_N = u_{N,m,r,\varphi}$ for u, the solution of equation (1.1), setting

$$u_N(x) = v_N(\varphi^{-1}(x))\,, \quad 0 \le x \le b. \tag{4.6}$$

Remark 2. The choice of nodes (4.2) with $\eta_1 = 0$, $\eta_m = 1$ in (4.5) actually implies that the resulting collocation approximation v_N belongs to the smoother spline space $S^{(0)}_{m-1}(\Delta^r_N)$ than it is stated by the condition (4.4).

Remark 3. The settings (4.4), (4.5) form a linear system of algebraic equations whose exact form is determined by the choice of a basis in $S^{(-1)}_{m-1}(\Delta^r_N)$. We refer to [13] for a convenient choice of it.

5 Convergence Results

Let X and Y be Banach spaces. By $\mathcal{L}(X,Y)$ we denote the Banach space of all linear continuous operators $A : X \to Y$ with the norm

$$\|A\|_{\mathcal{L}(X,Y)} = \sup\{\|Az\|_Y :\ z \in X,\ \ \|z\|_X \le 1\}.$$

By $C[a,b]$ we denote the Banach space of continuous functions z on $[a,b]$ with the usual norm $\|z\| = \max\{|z(t)| : t \in [a,b]\}$.

Theorem 1. *Let* $f \in C[0,b]$ *and* $K \in W^{0,\nu,\lambda}(D_b)$, $0<\nu<1$, $0 \le \lambda < 1-\nu$. *Furthermore, assume that* φ *is the transformation* (3.1) *and the interpolation nodes* (4.2) *with grid points* (1.2) *and parameters* (4.3) *are used. Then equation* (1.1) *has a unique solution* $u \in C[0,b]$, *the settings* (4.4)–(4.6) *determine for sufficiently large* N *a unique approximation* u_N *for* u *and*

$$\|u_N - u\|_\infty \to 0 \quad as \quad N \to \infty\,, \tag{5.1}$$

where $\|u_N - u\|_\infty = \sup\limits_{0 \le x \le b} |u_N(x) - u(x)|$.

Proof. We write (4.1) in the form $u_\varphi = T_\varphi u_\varphi + f_\varphi$ where T_φ is defined by formula

$$(T_\varphi z)(t) = \int_0^t K_\varphi(t,s) z(s)\, ds, \quad 0 \le t \le b.$$

It follows from (1.3) and (3.1) that $K_\varphi(t,s)$ is continuous in D_b and

$$|K_\varphi(t,s)| \le c(t-s)^{-\nu} s^{-\lambda}, \quad (t,s) \in D_b.$$

Since $\nu + \lambda < 1$, T_φ is compact as an operator from $L^\infty(0,b)$ into $C[0,b]$, see [14]. This together with $f_\varphi \in C[0,b]$ yields that equation $u_\varphi = T_\varphi u_\varphi + f_\varphi$ (equation (4.1)) has a unique solution $u_\varphi \in C[0,b]$. In particular, (1.1) has a unique solution $u \in C[0,b]$.

Further, conditions (4.4), (4.5) have the operator equation representation

$$v_N = P_N T_\varphi v_N + P_N f_\varphi, \tag{5.2}$$

where P_N is an operator which assigns to every continuous function $z \in C[0,b]$ its piecewise polynomial function $P_N z \in S_{m-1}^{(-1)}(\varDelta_N^r)$ such that $(P_N z)(x_{jl}) = z(x_{jl})$, $l = 1,\dots,m$; $j = 1,\dots,N$. It follows from [17] that the norms of $P_N \in \mathcal{L}(C[0,b], L^\infty(0,b))$ are bounded by a constant c which is independent of N,

$$\|P_N\|_{\mathcal{L}(C[0,b],L^\infty(0,b))} \le c, \tag{5.3}$$

and

$$\|z - P_N z\|_\infty \to 0 \quad \text{as} \quad N \to \infty \quad \text{for every} \quad z \in C[0,b]. \tag{5.4}$$

Using a standard argumentation (cf. [13, 15, 17]) we obtain that equation (5.2) has for sufficiently large values of N, say $N \ge N_0$, a unique solution $v_N \in S_{m-1}^{(-1)}(\varDelta_N^r)$ and

$$\|v_N - u_\varphi\|_\infty \le c\|u_\varphi - P_N u_\varphi\|_\infty, \quad N \ge N_0. \tag{5.5}$$

Here u_φ is the solution of equation (4.1) and c is a positive constant not depending on N. Since $u_\varphi \in C[0,b]$, we get from (5.4) and (5.5) that $\|v_N - u_\varphi\|_\infty \to 0$ as $N \to \infty$. This together with

$$\|u_N - u\|_\infty = \|v_N - u_\varphi\|_\infty \tag{5.6}$$

yields (5.1). □

Next we establish a global convergence result for method (4.4)–(4.6).

Theorem 2. *Let the following conditions be fulfilled:*

1. $K \in W^{m,\nu,\lambda}(D_b)$, $f \in C^{m,\nu+\lambda}(0,b]$, $m \in \mathbb{N}$, $0 < \nu < 1$, $0 \le \lambda < 1-\nu$;
2. φ *is the transformation* (3.1)*;*
3. *the interpolation nodes* (4.2) *with grid points* (1.2) *and parameters* (4.3) *are used.*

Then the settings (4.4)–(4.6) *determine for* $N \ge N_0$ *a unique approximation* u_N *to* u*, the solution to* (1.1)*, and*

$$\|u_N - u\|_\infty \le c \begin{cases} N^{-r\varrho(1-\nu-\lambda)} & \text{for } 1 \le r < \frac{m}{\varrho(1-\nu-\lambda)}, \\ N^{-m} & \text{for } r \ge \frac{m}{\varrho(1-\nu-\lambda)},\ r \ge 1, \end{cases} \tag{5.7}$$

where c *is a positive constant not depending on* N*.*

Proof. On the basis of Lemmas 1 and 2 we find that $u_\varphi \in C[0,b] \cap C^m(0,b]$ and for every $s \in (0,b]$ and $j = 1,\dots,m$,

$$|u_\varphi^{(j)}(s)| \le c \begin{cases} 1 & \text{if} \quad j \le \varrho(1-\nu-\lambda), \\ s^{\varrho(1-\nu-\lambda)-j} & \text{if} \quad j > \varrho(1-\nu-\lambda). \end{cases} \tag{5.8}$$

For a spline $w_N \in S_{m-1}^{(-1)}(\Delta_N^r)$ denote $w_{N,j} = w_N\big|_{[x_{j-1},x_j]}$, $j = 1,\dots,N$. Due to (5.3) we get the estimate

$$\begin{aligned} \|u_\varphi - P_N u_\varphi\|_\infty &= \|u_\varphi - w_N - P_N(u_\varphi - w_N)\|_\infty \\ &\le c \max_{j=1,\dots,N} \max_{x_{j-1} \le x \le x_j} |u_\varphi(x) - w_{N,j}(x)|, \end{aligned} \tag{5.9}$$

with a positive constant c which is independent of N. We fix $w_{N,j}$ as a Taylor polynomial for $u_\varphi(x)$ at $x = x_j$:

$$w_{N,j}(x) = \sum_{k=0}^{m-1} \frac{u_\varphi^{(k)}(x_j)}{k!}(x - x_j)^k, \quad x_{j-1} \le x \le x_j\,.$$

The integral form of the reminder term of the $(m-1)th$ order Taylor approximation of $u_\varphi(x)$ at $x = x_j$ and the estimate (5.8) gives us for all $x \in [x_{j-1}, x_j]$ $(j = 1,\dots,N)$ the inequality

$$|u_\varphi(x) - w_{N,j}(x)| \le c \int_x^{x_j} (s-x)^{m-1} \left\{ \begin{array}{ll} 1 & \text{if} \quad m \le \varrho(1-\nu-\lambda) \\ s^{\varrho(1-\nu-\lambda)-m} & \text{if} \quad m > \varrho(1-\nu-\lambda) \end{array} \right\} ds. \tag{5.10}$$

Due to (1.2),

$$x_j - x_{j-1} \le brN^{-1}, \quad j = 1,\dots,N. \tag{5.11}$$

If $m \le \varrho(1-\nu-\lambda)$, then we obtain from (5.10) and (5.11) that

$$|u_\varphi(x) - w_{N,j}(x)| \le cN^{-m}, \quad x_{j-1} \le x \le x_j\,, \quad j = 1,\dots,N, \tag{5.12}$$

where c is a positive constant not depending on N.

In the case $m > \varrho(1-\nu-\lambda)$ we have

$$\max_{0\le x\le x_1}\int_x^{x_1}(s-x)^{m-1}s^{\varrho(1-\nu-\lambda)-m}ds \le \max_{0\le x\le x_1}\int_x^{x_1}(s-x)^{\varrho(1-\nu-\lambda)-1}ds$$

$$\le c_1\begin{cases} N^{-\varrho r(1-\nu-\lambda)} & \text{for} \quad 1\le r<\frac{m}{\varrho(1-\nu-\lambda)},\\ N^{-m} & \text{for} \quad r\ge \frac{m}{\varrho(1-\nu-\lambda)},\ r\ge 1, \end{cases} \tag{5.13}$$

$$\max_{j=2,\ldots,N}\max_{x_{j-1}\le x\le x_j}\int_x^{x_j}(s-x)^{m-1}s^{\varrho(1-\nu-\lambda)-m}ds$$

$$\le \max_{j=2,\ldots,N}\max_{x_{j-1}\le x\le x_j} x^{\varrho(1-\nu-\lambda)-m}\int_x^{x_j}(s-x)^{m-1}ds$$

$$\le c_2\begin{cases} N^{-\varrho r(1-\nu-\lambda)} & \text{for} \quad 1\le r<\frac{m}{\varrho(1-\nu-\lambda)},\\ N^{-m} & \text{for} \quad r\ge \frac{m}{\varrho(1-\nu-\lambda)},\ r\ge 1, \end{cases} \tag{5.14}$$

where c_1 and c_2 are some positive constants not depending on N. It follows from (5.9), (5.10) and (5.12)–(5.14) that

$$\|u_\varphi - P_N u_\varphi\|_\infty \le c\begin{cases} N^{-r\varrho(1-\nu-\lambda)} & \text{for } 1\le r<\frac{m}{\varrho(1-\nu-\lambda)},\\ N^{-m} & \text{for } r\ge \frac{m}{\varrho(1-\nu-\lambda)},\ r\ge 1, \end{cases}$$

with a positive constant c which is independent of N. This together with (5.5) and (5.6) yields (5.7). □

Remark 4. It follows from Theorem 2 that the accuracy $\|u_N - u\|_\infty \le cN^{-m}$ can be achieved on a mildly graded or uniform grid. As an example, if we assume that $\nu = 2/5$, $\lambda = 1/5$, $m = 3$ (the case of piecewise quadratic polynomials), $\varrho \ge 15/2$, the maximal convergence order $\|u_N - u\|_\infty \le cN^{-3}$ is available for $r \ge 1$. In particular, the uniform grid with nodes (1.2), $r = 1$, may be used.

Remark 5. In addition to Theorem 2, assuming some additional smoothness of f and g (see (1.3)) and choosing more carefully the collocation parameters (4.3), the superconvergence of v_N at the collocation points (4.2) can be established, cf. [1, 3, 4, 5, 13, 17]. More precisely, let $K \in W^{m+1,\nu,\lambda}(D_b)$, $f \in C^{m+1,\nu+\lambda}(0,b]$, $m \in \mathbb{N}$, $0<\nu<1$, $0\le\lambda<1-\nu$, and let the interpolation nodes (4.2) be generated by the grid points (1.2) and by the node points $\eta_1,\ldots,\eta_m$ of a quadrature approximation

$$\int_0^1 z(s)ds \approx \sum_{l=1}^m w_l z(\eta_l), \quad 0\le\eta_1<\ldots<\eta_m\le 1, \tag{5.15}$$

which, with appropriate weights $\{w_l\}$, is exact for all polynomials of degree m.

Then it turns out that for sufficiently large N,

$$\max_{\substack{l=1,\dots,m;\\ j=1,\dots,N}} |u_N(\varphi(x_{jl})) - u(\varphi(x_{jl}))| = \max_{\substack{l=1,\dots,m;\\ j=1,\dots,N}} |v_N(x_{jl}) - u_\varphi(x_{jl})|$$
$$\leq c \begin{cases} N^{-2\varrho r(1-\nu-\lambda)} & \text{for} \quad 1 \leq \quad r < \frac{m+1-\nu}{2\varrho(1-\nu-\lambda)}, \\ N^{-m-(1-\nu)} & \text{for} \qquad\quad r \geq \frac{m+1-\nu}{2\varrho(1-\nu-\lambda)}, \quad r \geq 1. \end{cases} \tag{5.16}$$

We will investigate this question in a forthcoming paper where a more general class of integral equations with diagonal and boundary singularities will be discussed.

6 Numerical Example

Let us consider the following equation:

$$u(x) = \int_0^x (x-y)^{-\nu} y^{-\lambda} u(y)\, dy + f(x)\,, \quad 0 \leq x \leq 1, \tag{6.1}$$

where $0 < \nu < 1\,,\quad 0 \leq \lambda < 1,\ \nu + \lambda < 1$. The forcing function f is selected so that $u(x) = x^{1-\nu-\lambda}$ is the exact solution to (6.1). Actually, this is a problem of the form (1.1), (1.3) where $b = 1$, $g(x,y) \equiv 1$, $K(x,y) = (x-y)^{-\nu} y^{-\lambda}$,

$$f(x) = x^{1-\nu-\lambda} - x^{2(1-\nu-\lambda)} \frac{\Gamma(1-\nu)\,\Gamma(2(1-\lambda)-\nu)}{\Gamma(3-2(\nu+\lambda))}, \quad 0 \leq x \leq 1,$$

$$\Gamma(t) = \int_0^\infty e^{-s} s^{t-1}\, ds\,, \quad t > 0\,.$$

It is easy to check that in this case $K \in W^{m,\nu,\lambda}(D_1)$ and $f \in C^{m,\nu+\lambda}(0,1]$ for arbitrary $m \in \mathbb{N}$.

Equation (6.1) was solved numerically by method (4.4)–(4.6) for $\nu = 2/5$, $\lambda = 1/5$, $m = 3$, $\eta_1 = (5-\sqrt{15})/10$, $\eta_2 = 1/2$, $\eta_3 = (5+\sqrt{15})/10$. Here η_1, η_2, η_3 are the node points of the Gauss-Legendre quadrature rule (5.15) by $m = 3$. This formula is exact for all polynomials of degree not exceeding $2m - 1 = 5$.

In Tables 1 and 2 some results for different values of the parameters N, ϱ and r are presented. The quantities $\varepsilon_N^{(\varrho,r)}$ in Table 1 are approximate values of the norm $\|u_N - u\|_\infty$, calculated as follows:

$$\varepsilon_N^{(\varrho,r)} = \max_{\substack{l=0,\dots,10\\ j=1,\dots,N}} |u_N((\tau_{jl}^{(r)})^\varrho) - u((\tau_{jl}^{(r)})^\varrho)|\,,$$

where $\tau_{jl}^{(r)} = x_{j-1} + l(x_j - x_{j-1})/10, \quad l = 0,\dots,10; \quad j = 1,\dots,N$, with the grid points x_j, defined by formula (1.2) for $b = 1$.

Table 2 shows the dependence of

$$\gamma_N^{(\varrho,r)} = \max_{\substack{l=1,\dots,m;\\ j=1,\dots,N}} |u_N(\varphi(x_{jl})) - u(\varphi(x_{jl}))| = \max_{\substack{l=1,\dots,m;\\ j=1,\dots,N}} |v_N(x_{jl}) - u_\varphi(x_{jl})|$$

on the parameters N, ϱ and r (see (5.16)). The ratios $\delta_N^{(\varrho,r)} = \varepsilon_{N/2}^{(\varrho,r)} \Big/ \varepsilon_N^{(\varrho,r)}$, $\widetilde{\delta}_N^{(\varrho,r)} = \gamma_{N/2}^{(\varrho,r)} \Big/ \gamma_N^{(\varrho,r)}$, characterizing the observed convergence rate, are also presented. From Theorem 2 it follows that for sufficiently large N,

$$\varepsilon_N^{(\varrho,r)} \approx \|u_N - u\|_\infty \le c \begin{cases} N^{-2\varrho r/5} & \text{if} \quad 1 \le \varrho r < 15/2, \\ N^{-3} & \text{if} \quad \varrho r \ge 15/2\,. \end{cases} \tag{6.2}$$

Table 1. ($m = 3, \nu = \frac{2}{5}, \lambda = \frac{1}{5}, \eta_1 = \frac{5-\sqrt{15}}{10}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5+\sqrt{15}}{10}$)

N	$\varepsilon_N^{(1,1)}$ $\delta_N^{(1,1)}$	$\varepsilon_N^{(3,1)}$ $\delta_N^{(3,1)}$	$\varepsilon_N^{(7/2,\,3/2)}$ $\delta_N^{(7/2,\,3/2)}$	$\varepsilon_N^{(15/2,\,1)}$ $\delta_N^{(15/2,\,1)}$	$\varepsilon_N^{(15/4,\,2)}$ $\delta_N^{(15/4,\,2)}$
32	7.7 E - 2 1.35	3.1 E - 4 2.30	1.7 E - 5 4.29	1.8 E - 6 8.64	9.2 E - 7 8.75
64	5.8 E - 2 1.34	1.3 E - 4 2.30	3.9 E - 6 4.29	2.1 E - 7 8.47	1.1 E - 7 8.05
128	4.3 E - 2 1.33	5.8 E - 5 2.30	9.1 E - 7 4.29	2.6 E - 8 8.32	1.4 E - 8 8.00
256	3.2 E - 2 1.33	2.5 E - 5 2.30	2.1 E - 7 4.29	3.1 E - 9 8.22	1.8 E - 9 8.00
512	2.4 E - 2 1.33	1.1 E - 5 2.30	4.9 E - 8 4.29	3.9 E - 10 8.14	2.2 E - 10 8.00
	1.33	2.30	4.29	8.00	8.00

Due to (6.2), the ratio $\delta_N^{(\varrho,r)}$ ought to be approximately

$$(N/2)^{-2\varrho r/5}/N^{-2\varrho r/5} = 2^{2\varrho r/5} \ \text{ for } \ 1 \le \varrho r < \frac{15}{2}$$

and 8 for $\varrho r \ge 15/2$. In particular, $\delta_N^{(1,1)}$, $\delta_N^{(3,1)}$, $\delta_N^{(\frac{7}{2},\frac{3}{2})}$, $\delta_N^{(\frac{15}{2},1)}$ and $\delta_N^{(\frac{15}{4},2)}$ ought to be approximately 1.33, 2.30, 4.29, 8.00 and 8.00, respectively. These values of $\delta_N^{(\varrho,r)}$ are given in the last row of Table 1.

Table 2. ($m = 3, \nu = \frac{2}{5}, \lambda = \frac{1}{5}, \eta_1 = \frac{5-\sqrt{15}}{10}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5+\sqrt{15}}{10}$)

N	$\gamma_N^{(1,1)}$ $\widetilde{\delta}_N^{(1,1)}$	$\gamma_N^{(3,1)}$ $\widetilde{\delta}_N^{(3,1)}$	$\gamma_N^{(4,1)}$ $\widetilde{\delta}_N^{(4,1)}$	$\gamma_N^{(3,\,3/2)}$ $\widetilde{\delta}_N^{(3,\,3/2)}$	$\gamma_N^{(4,2)}$ $\widetilde{\delta}_N^{(4,2)}$
32	1.4 E - 2 2.30	3.5 E - 7 5.34	3.8 E - 8 9.23	1.7 E - 8 11.74	8.0 E - 8 11.91
64	6.3 E - 3 2.30	6.5 E - 8 5.28	4.2 E - 9 9.20	1.4 E - 9 11.95	6.8 E - 9 11.81
128	2.7 E - 3 2.30	1.2 E - 8 5.28	4.5 E - 10 9.19	1.2 E - 10 12.04	5.7 E - 10 11.92
256	1.2 E - 3 2.30	2.4 E - 9 5.28	4.9 E - 11 9.19	9.6 E - 12 12.08	4.7 E - 11 12.01
512	5.2 E - 4 2.30	4.5 E - 10 5.28	5.4 E - 12 9.19	7.9 E - 13 12.10	3.9 E - 12 12.06
	1.74	5.28	9.19	12.13	12.13

In a similar way we obtain from (5.16) that $\widetilde{\delta}_N^{(1,1)}$, $\widetilde{\delta}_N^{(3,1)}$,$\widetilde{\delta}_N^{(4,1)}$, $\widetilde{\delta}_N^{(3,\frac{3}{2})}$ and $\widetilde{\delta}_N^{(4,2)}$ ought to be approximately 1.74, 5.28, 9.19, 12.13 and 12.13, respectively. These values of $\widetilde{\delta}_N^{(\varrho,r)}$ are given in the last row of Table 2.

As we can see from Tables 1 and 2, the numerical results are in good agreement with the theoretical estimates. In Table 2 only the decrease of $\gamma_N^{(1,1)}$ is faster than it is indicated by theoretical estimates: the predicted value for $\widetilde{\delta}_N^{(1,1)}$ is equal to 1.74, but the current experiment gave for $\widetilde{\delta}_N^{(1,1)}$ a stable value 2.30. This phenomenon notifies that the local order of convergence of proposed algorithms needs further theoretical and numerical study.

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