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A Variational Approach for an Electro-Elastic Unilateral Contact Problem*

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Abstract. We consider a mechanical model which describes the frictionless unilateral contact between an electro-elastic body and a rigid electrically non-conductive foundation. For this model, a mixed variational formulation is provided. Then, using elements of the saddle point theory and a fixed point technique, an abstract result is proved. Based on this result, the existence of a unique weak solution of the mechanical problem is established.

Key words: piezoelectricity, frictionless unilateral contact, mixed variational formulation, weak solution.

1 Introduction

In this paper we study a frictionless unilateral contact problem involving the piezoelectric effect. The piezoelectricity can be described as follows: when mechanical pressure is applied to a certain classes of crystalline materials (e.g ceramics $BaTiO_3$, $BiFeO_3$), the crystalline structure produces a voltage proportional to the pressure. Conversely, when an electric field is applied, the structure changes his shape producing dimensional modifications in the material. Actually, there is a big interest into the study of piezoelectric materials, this type of materials being used in radioelectronics, electroacoustics and measuring equipments. In the same time, due to the fact that the parts of the equipments are in contact, the interest for the contact problems is increasing. The literature concerning this topic is very rich, see for example [10, 11, 14, 15, 16, 22, 23, 24] for modelling in piezoelectricity and [5, 6, 13, 17, 20, 25] for the modelling and the analysis of the contact processes. Some theoretical results for contact models taking into account the interaction between the electric and the mechanic fields have been obtained in [7, 18, 19, 21].

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In this paper we use a mixed variational formulation of the piezoelectric contact problem, which consists one of the traits of the novelty of the present paper. For details concerning the mathematical tools we refer to [1, 2, 3, 4, 6, 12]. The main motivation of this approach is that in the last years this type of formulation in contact mechanics is preferred from numerical point of view, see for example [7, 8, 9]. The present paper follows [7] where a frictional bilateral contact problem for electro-elastic materials was treated.

The rest of this paper is organized as follows. In Section 2 we provide some notation and preliminaries. In Section 3 we describe the physical setting, formulate the mathematical problem and state the main result, Theorem 1. In Section 4 we provide an abstract result, Theorem 2, then we prove Theorem 1. The last section contains some conclusions and comments.

2 Notation and Preliminaries

Let us denote by $\mathbb{S}^3$ the space of second order symmetric tensors on $\mathbb{R}^3$. Every element in $\mathbb{R}^3$ or $\mathbb{S}^3$ will be typeset in boldface and by " $\cdot$ " and $|\cdot|$ we denote the inner product and the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}^3$, respectively. Thus,

$$
\begin{aligned}
&\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i, \quad |\boldsymbol{v}|=(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}, \quad \boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\tau}|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2},\\
&\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^3, \quad \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^3.
\end{aligned}
$$

Here and below, the indices i and j run between 1 and 3 and the summation convention over repeated indices is adopted.

Let $\Omega\subset\mathbb{R}^3$ be a bounded domain. We introduce the following functional spaces on Ω,

$$
\begin{aligned}
&H=\{\boldsymbol{u}=(u_i)\mid\ u_i\in L^2(\Omega)\}, \quad \mathcal{H}=\{\boldsymbol{\sigma}=(\sigma_{ij})\mid\ \sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\},\\
&H_1=\{\boldsymbol{u}\in H\mid\ \boldsymbol{\varepsilon}(\boldsymbol{u})\in\mathcal{H}\}, \quad \mathcal{H}_1=\{\boldsymbol{\sigma}\in\mathcal{H}\mid\ \operatorname{Div}\boldsymbol{\sigma}\in H\},
\end{aligned}
$$

where

$$
\boldsymbol{\varepsilon}(\boldsymbol{u})=(\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u})=\frac{1}{2}(u_{i,j}+u_{j,i}), \quad \operatorname{Div}\boldsymbol{\sigma}=(\sigma_{ij,j}).
$$

Here and below the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products,

$$
\begin{aligned}
(\boldsymbol{u},\boldsymbol{v})_H&=\int_\Omega u_iv_i\,dx, \qquad (\boldsymbol{u},\boldsymbol{v})_{H_1}=(\boldsymbol{u},\boldsymbol{v})_H+(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},\\
(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}&=\int_\Omega \sigma_{ij}\tau_{ij}\,dx, \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1}=(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}+(\operatorname{Div}\boldsymbol{\sigma},\operatorname{Div}\boldsymbol{\tau})_H.
\end{aligned}
$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

Let us assume that the boundary of Ω, denoted by Γ is Lipschitz continuous. We denote by $\boldsymbol{n}$ the unit outward normal vector on the boundary, defined a.e. In order to simplify the writing, everywhere below, for every field (scalar, vectorial or tensorial) we will use the same notation in order to indicate his Sobolev trace on Γ.

For a vectorial field $\boldsymbol{v}$, we denote by v_n and $\boldsymbol{v}_\tau$ the *normal* and the *tangential* components on the boundary, defined as follows:

$$v_n = \boldsymbol{v} \cdot \boldsymbol{n}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_n \boldsymbol{n}.$$

For a regular (say C^1) stress field $\boldsymbol{\sigma}$, the application of its trace on the boundary to $\boldsymbol{n}$ is the Cauchy stress vector $\boldsymbol{\sigma n}$. Furthermore, we define the *normal* and *tangential* components of the Cauchy vector on the boundary by the formulas

$$\sigma_n = (\boldsymbol{\sigma n}) \cdot \boldsymbol{n}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma n} - \sigma_n \boldsymbol{n}$$

and we note that the following identity takes place

$$\boldsymbol{\sigma n} \cdot \boldsymbol{v} = \sigma_n v_n + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau. \tag{2.1}$$

Finally, we recall the useful Green's formula

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\, \boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma n} \cdot \boldsymbol{v}\, da \quad \forall \boldsymbol{v} \in H_1. \tag{2.2}$$

For a proof of the formula (2.2) and for more details related to this section, we send the reader to [5].

3 The Model and the Statement of the Main Result

We consider an elasto-piezoelectric body that occupies the bounded domain $\Omega \subset \mathbb{R}^3$, in contact with a rigid electrically non-conductive foundation. We assume that the boundary Γ is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $\mathrm{meas}(\Gamma_1) > 0$ and $\overline{\Gamma}_3$ is a compact subset of $\Gamma \backslash \overline{\Gamma}_1$. Let us denote by $\boldsymbol{n}_3$ the restriction of $\boldsymbol{n}$ to Γ_3. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface traction of density $\boldsymbol{f}_2$ act on Γ_2. Moreover, we assume that Γ_3 is the potential contact zone and we denote by $g : \Gamma_3 \to \mathbb{R}$ the gap function. By gap in a given point of Γ_3 we understand the distance between the deformable body and the foundation measured along of the outward normal $\boldsymbol{n}$. Let us consider a second partition of the boundary Γ in two disjoint measurable parts Γ_a and Γ_b such that $\mathrm{meas}(\Gamma_a) > 0$ and $\Gamma_b \supseteq \Gamma_3$. On Γ_a the electrical potential vanishes and on Γ_b we assume electric charges of density q_2. Since the foundation is electrically non-conductive, and assuming that the gap zone is also electrically non-conductive, q_2 must vanish on Γ_3. By q_0 we will denote the density of the free electric charges on Ω.

We denote by $\boldsymbol{u} = (u_i)$ the displacement field, by $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})$ the stress tensor, by φ the electric potential field and by $\boldsymbol{D} = (D_i)$ the electrical field. We start the modelling writing the universal equilibrium equations

$$\mathrm{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\mathrm{div}\, \boldsymbol{D} = \boldsymbol{q}_0 \qquad \text{in} \quad \Omega. \tag{3.2}$$

In order to describe the behavior of the materials, we use the constitutive law

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{E}^\top \nabla\varphi \quad \text{in} \quad \Omega, \tag{3.3}$$
$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\beta}\nabla\varphi \quad \text{in} \quad \Omega, \tag{3.4}$$

where $\mathcal{C} = (\mathcal{C}_{ijls})$ is the elasticity tensor, $\mathcal{E} = (\mathcal{E}_{ijl})$ is the piezoelectric tensor and $\boldsymbol{\beta}$ is the permitivity tensor. We use here $\mathcal{E}^\top$ to denote the transpose of the tensor $\mathcal{E}$,

$$\mathcal{E}\,\boldsymbol{\sigma}\cdot\boldsymbol{v} = \boldsymbol{\sigma}\cdot\mathcal{E}^\top\boldsymbol{v}, \qquad \forall\boldsymbol{\sigma}\in S^3,\ \boldsymbol{v}\in\mathbb{R}^3,$$

and we notice that $\mathcal{E}^\top = (\mathcal{E}^\top_{ijl}) = (\mathcal{E}_{lij})$ for all $i, j, l \in \overline{1,3}$. Such kind of electromechanic relations can be found in the literature, see, [24].

To complete the model, we prescribe the mechanical and the electrical boundary conditions. According to the physical setting we write

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \quad \boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{3.5}$$
$$\varphi = 0 \quad \text{on} \quad \Gamma_a, \quad \boldsymbol{D}\cdot\boldsymbol{n} = q_2 \quad \text{on} \quad \Gamma_b. \tag{3.6}$$

To model the contact process, we use the Signorini condition with non-zero gap. In addition, we assume that the contact is frictionless. Consequently, we can express mathematically the frictionless contact condition as follows

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0}, \quad \sigma_n \le 0, \quad u_n \le g, \quad \sigma_n(u_n - g) = 0 \quad \text{on } \Gamma_3. \tag{3.7}$$

Knowing the displacement field $\boldsymbol{u}$ and the electric field φ we can compute the stress tensor $\boldsymbol{\sigma}$ and the electric displacement $\boldsymbol{D}$ using (3.3) and (3.4), respectively. Therefore, the displacement field $\boldsymbol{u}$ and the electric field φ are called the main unknowns.

To resume, we consider the following problem.

Problem 1. Find the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and the electric potential field $\varphi : \Omega \to \mathbb{R}$ such that (3.1)–(3.7) hold.

In order to study this problem, we make the following assumptions:

- For

$$\mathcal{C} = (\mathcal{C}_{ijls}) : \ \Omega \times \mathbb{S}^3 \to \mathbb{S}^3, \quad \mathcal{C}_{ijls} = \mathcal{C}_{ijsl} = \mathcal{C}_{lsij} \in L^\infty(\Omega) \tag{3.8}$$

  there exists $m_\mathcal{C} > 0$ such that $\mathcal{C}_{ijls}\varepsilon_{ij}\varepsilon_{ls} \ge m_\mathcal{C}\,|\boldsymbol{\varepsilon}|^2$, $\forall\boldsymbol{\varepsilon}\in\mathbb{S}^3$, a.e. on Ω, where

$$\mathcal{E} = (\mathcal{E}_{ijk}) \ : \ \Omega \times \mathbb{S}^3 \to \mathbb{R}^3, \quad \mathcal{E}_{ijk} = \mathcal{E}_{ikj} \in L^\infty\Omega). \tag{3.9}$$

- For the permitivity tensor

$$\boldsymbol{\beta} = (\boldsymbol{\beta}_{ij}) \ : \ \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \boldsymbol{\beta}_{ij} = \boldsymbol{\beta}_{ji} \in L^\infty(\Omega) \tag{3.10}$$

  there exists $m_\beta > 0$ such that $\boldsymbol{\beta}_{ij}E_iE_j \ge m_\beta|E|^2$, $\forall\, E \in \mathbb{R}^3$, a.e. on Ω.

Concerning the mechanical and the electrical data we will assume

$$\boldsymbol{f}_0 \in L^2(\Omega)^3, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3, \quad q_0 \in L^2(\Omega), \quad q_2 \in L^2(\Gamma_b). \tag{3.11}$$

Moreover, there exists $g_{ext} : \Omega \to \mathbb{R}$ such that

$$g_{ext} \in H^1(\Omega), \;\; g_{ext} = 0 \text{ on } \Gamma_1, \;\; g_{ext} \geq 0 \text{ on } \Gamma \setminus \Gamma_1, \;\; g = g_{ext} \text{ on } \Gamma_3, \tag{3.12}$$

$$\text{the unit outward normal to } \Gamma_3, \; \boldsymbol{n}_3, \text{ is assumed to be constant.} \tag{3.13}$$

Based on these assumptions, we will give a mixed variational formulation of this mechanical problem, using the Hilbert spaces,

$$V = \{\, \boldsymbol{v} \in H_1 | \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}, \; \mathcal{W} = \{\psi \in H^1(\Omega) | \psi = 0 \text{ on } \Gamma_a\}, \; \tilde{V} = V \times \mathcal{W}.$$

We consider the inner products $(\cdot,\cdot)_V : V \times V \to \mathbb{R}$, $(\cdot,\cdot)_{\mathcal{W}} : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ and $(\cdot,\cdot)_{\tilde{V}} : \tilde{V} \times \tilde{V} \to \mathbb{R}$ defined as follows

$$\begin{aligned} &(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad (\varphi,\psi)_{\mathcal{W}} = (\nabla\varphi,\nabla\psi)_H, \\ &(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}})_{\tilde{V}} = (\boldsymbol{u},\boldsymbol{v})_V + (\varphi,\psi)_{\mathcal{W}}. \end{aligned} \tag{3.14}$$

Let us remember the Korn inequality: there exists $c_K = c_K(\Omega,\Gamma_1) > 0$ such that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \geq c_K \|\boldsymbol{v}\|_{H_1}, \qquad \forall \boldsymbol{v} \in V.$$

Furthermore, the following Poincaré's type inequality takes place: there exists $c_P = c_P(\Omega,\Gamma_a) > 0$ such that

$$\|\varphi\|_{L^2(\Omega)} \leq c_P \|\nabla\varphi\|_{[L^2(\Omega)]^3}, \;\; \forall \varphi \in \mathcal{W}.$$

Consequently $(V,(\cdot,\cdot)_V,\|\cdot\|_V)$, $(\mathcal{W},(\cdot,\cdot)_{\mathcal{W}},\|\cdot\|_{\mathcal{W}})$, and $(\tilde{V},(\cdot,\cdot)_{\tilde{V}},\|\cdot\|_{\tilde{V}})$, are Hilbert spaces.

Keeping in mind (3.1) and (3.3), the Green's formula (2.2) yields for all $\boldsymbol{v} \in V$

$$\int_\Omega \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega \mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \nabla\varphi\,dx = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,dx + \int_\Gamma \boldsymbol{\sigma n} \cdot \boldsymbol{v}\,da. \tag{3.15}$$

Next, multiplying (3.4) by $\nabla\psi$, for all $\psi \in \mathcal{W}$, we deduce

$$\int_\Omega \mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \nabla\psi\,dx - \int_\Omega \boldsymbol{\beta}\,\nabla\varphi \cdot \nabla\psi\,dx = \int_{\partial\Omega} \boldsymbol{D} \cdot \boldsymbol{n} - \int_\Omega \operatorname{div} \boldsymbol{D}\,\psi. \tag{3.16}$$

Subtracting (3.16) from (3.15) and keeping in mind (3.2), (3.5) and (3.6), we obtain, for every $(\boldsymbol{v},\psi) \in \tilde{V}$,

$$\begin{aligned} &\int_\Omega \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega \mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \nabla\varphi\,dx - \int_\Omega \mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \nabla\psi\,dx + \int_\Omega \boldsymbol{\beta}\,\nabla\varphi \cdot \nabla\psi\,dx \\ &\quad = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\,da - \int_{\Gamma_b} q_2\,\psi\,da + \int_\Omega q_0\,\psi\,dx + \int_{\Gamma_3} \boldsymbol{\sigma n} \cdot \boldsymbol{v}\,da. \end{aligned}$$

Since $\boldsymbol{\sigma}_\tau = \mathbf{0}$ on Γ_3, taking into account (2.1) we can write

$$\int_{\Gamma_3} \boldsymbol{\sigma n} \cdot \boldsymbol{v}\, da = \int_{\Gamma_3} \sigma_n\, v_n\, da, \quad \forall \boldsymbol{v} \in V.$$

Let us consider $a : \tilde{V} \times \tilde{V} \to \mathbb{R}$ the bilinear form,

$$\begin{aligned} a(\tilde{\boldsymbol{u}},\, \tilde{\boldsymbol{v}}) := &\int_\Omega \mathcal{C}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \nabla\varphi\, dx \\ &- \int_\Omega \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \nabla\psi\, dx + \int_\Omega \boldsymbol{\beta}\, \nabla\varphi \cdot \nabla\psi\, dx \end{aligned}$$

for all $\tilde{\boldsymbol{u}} = (\boldsymbol{u}, \varphi)$, $\tilde{\boldsymbol{v}} = (\boldsymbol{v}, \psi) \in \tilde{V}$. Moreover, using the Riesz representation theorem, we define $\tilde{\boldsymbol{f}} \in \tilde{V}$ such that for all $\tilde{\boldsymbol{v}} = (\boldsymbol{v}, \psi) \in \tilde{V}$,

$$(\tilde{\boldsymbol{f}},\, \tilde{\boldsymbol{v}})_{\tilde{V}} = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, da - \int_{\Gamma_b} q_2\, \psi\, da + \int_\Omega q_0\, \psi\, dx.$$

Consequently, using the Green formula (2.2), we deduce that

$$a(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) = (\tilde{\boldsymbol{f}},\, \tilde{\boldsymbol{v}})_{\tilde{V}} + \int_{\Gamma_3} \sigma_n v_n, \qquad \forall \tilde{\boldsymbol{v}} \in \tilde{V}. \tag{3.17}$$

In order to provide a mixed weak formulation, we define a dual Lagrange multiplier $\boldsymbol{\lambda} \in D := \left([H^{1/2}(\Gamma_3)]^3\right)'$ such that

$$\langle \boldsymbol{\lambda}, \boldsymbol{v} \rangle_{\Gamma_3} := -\int_{\Gamma_3} \sigma_n\, v_n ds, \qquad \forall \boldsymbol{v} \in [H^{1/2}(\Gamma_3)]^3, \tag{3.18}$$

where $[H^{1/2}(\Gamma_3)]^3$ denotes the space of restrictions of traces of all functions belonging to V and $\langle \cdot, \cdot \rangle_{\Gamma_3}$ denotes the duality pairing between D and $[H^{1/2}(\Gamma_3)]^3$. Moreover, we define a bilinear form $b : \tilde{V} \times D \to \mathbb{R}$, as follows

$$b(\tilde{\boldsymbol{v}},\, \boldsymbol{\mu}) := \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3}, \quad \forall \tilde{\boldsymbol{v}} = (\boldsymbol{v}, \psi) \in \tilde{V},\, \boldsymbol{\mu} \in D. \tag{3.19}$$

Using (3.19), keeping in mind that the Sobolev trace operator is linear and continuous and taking into account (3.14), we deduce that there exists $M_b > 0$ such that

$$|b(\tilde{\boldsymbol{v}},\, \boldsymbol{\mu})| \le M_b \|\tilde{\boldsymbol{v}}\|_{\tilde{V}} \|\boldsymbol{\mu}\|_D. \tag{3.20}$$

In addition, using the properties of the Sobolev trace operator, it can be shown that there exists $\alpha > 0$ such that

$$\inf_{\boldsymbol{\mu} \in D, \boldsymbol{\mu} \neq 0_D}\ \sup_{\tilde{\boldsymbol{v}} \in \tilde{V}, \tilde{\boldsymbol{v}} \neq 0_{\tilde{V}}} \frac{b(\tilde{\boldsymbol{v}}, \boldsymbol{\mu})}{\|\tilde{\boldsymbol{v}}\|_{\tilde{V}} \|\boldsymbol{\mu}\|_D} \ge \alpha. \tag{3.21}$$

Furthermore, we introduce a set as follows,

$$\Lambda := \left\{ \boldsymbol{\mu} \in D \,:\, \langle \boldsymbol{\mu},\, \boldsymbol{v} \rangle_{\Gamma_3} \le 0 \quad \forall \boldsymbol{v} \in K \right\}, \tag{3.22}$$

where

$$K := \Big\{ \boldsymbol{v} \in [H^{1/2}(\Gamma_3)]^3 \, : \, v_n \leq 0 \ \text{ on } \ \Gamma_3 \Big\}.$$

We note that $\boldsymbol{\lambda} \in \Lambda$ and, using the assumption (3.13), we deduce that $g_{ext}\boldsymbol{n}_3 \in V$, $g\boldsymbol{n}_3$ being the trace of $g_{ext}\boldsymbol{n}_3$ on Γ_3.

Taking into account the definition of $\boldsymbol{\lambda}$, (3.18), the definition of $b(\cdot,\cdot)$, (3.19), and the definition of Λ, (3.22), we get:

$$b(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) = \langle \boldsymbol{\lambda}, g\boldsymbol{n}_3 \rangle_{\Gamma_3}, \quad b(\tilde{\boldsymbol{u}}, \boldsymbol{\mu}) \leq \langle \boldsymbol{\mu}, g\boldsymbol{n}_3 \rangle_{\Gamma_3}, \quad \forall \boldsymbol{\mu} \in \Lambda.$$

Thus, denoting by $\tilde{\boldsymbol{g}}_{ext} := (g_{ext}\boldsymbol{n}_3, 0_{\mathcal{W}}) \in \tilde{V}$ we can write

$$b(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) = b(\tilde{\boldsymbol{g}}_{ext}, \boldsymbol{\lambda}), \quad b(\tilde{\boldsymbol{u}}, \boldsymbol{\mu}) \leq b(\tilde{\boldsymbol{g}}_{ext}, \boldsymbol{\mu}), \quad \forall \boldsymbol{\mu} \in \Lambda. \tag{3.23}$$

Keeping in mind (3.17), (3.23) we obtain the following weak formulation of Problem 1.

Problem 2. Find $\tilde{\boldsymbol{u}} \in \tilde{V}$ and $\boldsymbol{\lambda} \in \Lambda$, such that

$$\begin{aligned} &a(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) + b(\tilde{\boldsymbol{v}}, \boldsymbol{\lambda}) = (\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{v}})_V, && \forall \tilde{\boldsymbol{v}} \in V, \\ &b(\tilde{\boldsymbol{u}}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \leq b(\tilde{\boldsymbol{g}}_{ext}, \boldsymbol{\mu} - \boldsymbol{\lambda}), && \forall \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

The main result of this paper is the following.

Theorem 1. *Assume that* (3.8)–(3.13) *hold. Then, Problem 2 has a unique solution* $(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) \in \tilde{V} \times \Lambda$. *Moreover, if* $(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda})$ *and* $(\tilde{\boldsymbol{u}}^*, \boldsymbol{\lambda}^*)$ *are two solutions of Problem 2 corresponding to the data* $(\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}}_{ext}) \in \tilde{V} \times \tilde{V}$ *and* $(\tilde{\boldsymbol{f}}^*, \tilde{\boldsymbol{g}}^*_{ext}) \in \tilde{V} \times \tilde{V}$, *respectively, then*

$$\|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}^*\|_{\tilde{V}} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_D \leq C\big(\|\tilde{\boldsymbol{f}} - \tilde{\boldsymbol{f}}^*\|_{\tilde{V}} + \|\tilde{\boldsymbol{g}}_{ext} - \tilde{\boldsymbol{g}}^*_{ext}\|_{\tilde{V}}\big),$$

where $C = C(\mathcal{C}, \mathcal{E}, \boldsymbol{\beta}, \alpha, M_b) > 0$.

The proof of this theorem will be presented in the next section; it follows by using an abstract result, Theorem 2.

4 An Abstract Result and Proof of Theorem 1

Let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ be two Hilbert spaces and let us consider two bilinear forms as follows: $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ is non-symmetric and (a) there exists $M_a > 0$ such that

$$|a(u, v)| \leq M_a \|u\|_X \|v\|_X, \quad \forall u, v \in X, \tag{4.1}$$

(b) there exists $m_a > 0$ such that

$$a(v, v) \geq m_a \|v\|_X^2, \quad \forall v \in X, \tag{4.2}$$

and $b(\cdot,\cdot) : X \times Y \to \mathbb{R}$, for which (c) there exists $M_b > 0$ such that

$$|b(v, \mu)| \leq M_b \|v\|_X \|\mu\|_Y, \quad \forall v \in X, \, \mu \in Y, \tag{4.3}$$

(d) there exists $\alpha > 0$ such that

$$\inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha. \tag{4.4}$$

Let $\Lambda \subset Y$ be a closed, convex set that contains 0_Y. We consider now the following problem:

Problem 3. For given $f,\, g \in X$, find $u \in X$ and $\lambda \in \Lambda$ such that

$$a(u,v) + b(v,\lambda) = (f,\, v)_X, \qquad \forall v \in X, \tag{4.5}$$

$$b(u, \mu - \lambda) \leq b(g, \mu - \lambda), \qquad \forall \mu \in \Lambda. \tag{4.6}$$

We emphasize that the bilinear form $a(\cdot,\cdot)$ is *non-symmetric*. Consequently, Problem 3 is *not a saddle point problem*. Moreover, we are interested here in the case $g \neq 0_X$. An analysis of the particular case $g = 0_X$ can be found in [7].

The following result holds.

Theorem 2. *Let $f, g \in X$ and assume that* (4.1)–(4.4) *hold. Then, there exists a unique solution of Problem 3, $(u,\lambda) \in X \times \Lambda$. Moreover, if $(u_1,\, \lambda_1)$ and $(u_2,\, \lambda_2)$ are two solutions of Problem 3, corresponding to the data functions $f_1, g_1 \in X$ and $f_2, g_2 \in X$, then there exists $K = K(\alpha, m_a, M_a, M_b) > 0$ such that*

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \leq K(\|f_1 - f_2\|_X + \|g_1 - g_2\|_X). \tag{4.7}$$

Proof. Let $a_0(\cdot,\, \cdot)$ and $c(\cdot,\, \cdot)$ be the symmetric and the antisymmetric part of $a(\cdot,\, \cdot)$, respectively,

$$a_0 : X \times X \to \mathbb{R} \quad a_0(u,\, v) := \big(a(u,\, v) + a(v,\, u)\big)/2, \quad \forall u,\, v \in X,$$
$$c : X \times X \to \mathbb{R} \quad c(u,v) := \big(a(u,\, v) - a(v,\, u)\big)/2, \quad \forall u,\, v \in X.$$

For a given $r \in [0,1]$, we introduce the following bilinear form

$$a_r : X \times X \to \mathbb{R} \quad a_r\,(u,\, v) := a_0(u,\, v) + r\, c(u,\, v), \quad \forall u,\, v \in X. \tag{4.8}$$

We observe that for each $r \in [0,1]$,

$$a_r(u,u) \geq m_a \|u\|_X^2, \quad |a_r(u,v)| \leq 2M_a \|u\|_X \|v\|_X, \quad \forall u, v \in X.$$

Furthermore, we note that for all $u,\, v \in X$,

$$|a_0(u,v)| \leq M_a \|u\|_X \|v\|_X, \qquad |c(u,v)| \leq M_a \|u\|_X \|v\|_X, \quad \forall u, v \in X.$$

Let us consider the following auxiliary problem.

For given $f, g \in X$, find $u \in X$ and $\lambda \in \Lambda$, such that

$$a_r\,(u,v) + b(v,\lambda) = (f,\, v)_X, \qquad \forall v \in X, \tag{4.9}$$

$$b(u, \mu - \lambda) \leq b(g, \mu - \lambda), \qquad \forall \mu \in \Lambda. \tag{4.10}$$

The rest of the proof will be constructed in several steps.

Step 1. If $r = 0$, Problem (4.9)–(4.10) has a unique solution. Indeed, if $r = 0$, Problem (4.9)–(4.10) is equivalent to the saddle point problem: find $u \in X$ and $\lambda \in \Lambda$ such that

$$\mathcal{L}(u, \mu) \le \mathcal{L}(u, \lambda) \le \mathcal{L}(v, \lambda) \quad \forall v \in X, \, \mu \in \Lambda,$$

where $\mathcal{L} : X \times \Lambda \to \mathrm{I\!R}$ is the functional defined as follows:

$$\mathcal{L}(v, \mu) := \frac{1}{2} a(v, v) - (f, v)_X + b(v, \mu) - b(g, \mu).$$

According to [4], the previous saddle point problem has at least one solution. Consequently, Problem (4.9)–(4.10) has at least one solution. In fact, keeping in mind (4.4), (4.9) and (4.10) we deduce that Problem (4.9)–(4.10) has a unique solution $(u, \lambda) \in X \times \Lambda$. Indeed, let us consider (u^1, λ^1) and (u^2, λ^2) two solutions of Problem (4.9)–(4.10). Keeping in mind (4.9) we can write

$$a(u^1 - u^2, u^2 - u^1) + b(u^1 - u^2, \lambda^2 - \lambda^1) = 0. \tag{4.11}$$

Using now (4.10) we deduce that $b(u^1 - u^2, \lambda^2 - \lambda^1) \le 0$. Combining this inequality with (4.11) we obtain $u^1 = u^2$. Moreover,

$$b(v, \lambda^1 - \lambda^2) = -a(u^1 - u^2, v), \quad \forall v \in X.$$

Using now the inf-sup property (4.4) we find $\lambda^1 = \lambda^2$, that concludes Step 1.

Step 2. Let us assume that for given $f, g \in X$ there exists a unique solution of Problem (4.9)–(4.10), $(u, \lambda) \in X \times \Lambda$. If (u_1, λ_1) and (u_2, λ_2) are solutions of Problem (4.9)–(4.10) corresponding to two given data $(f_1, g) \in X \times X$ and $(f_2, g) \in X \times X$, respectively, then

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \le \frac{\alpha + m_a + 2\, M_a}{\alpha\, m_a} \, \|f_1 - f_2\|_X. \tag{4.12}$$

Indeed, using (4.9) and (4.10) we can write

$$\begin{aligned} &a_r\,(u_1 - u_2,\, u_1 - u_2) = (f_1 - f_2,\, u_1 - u_2)_X + b(u_1 - u_2, \lambda_2 - \lambda_1), \\ &b(u_1 - u_2, \lambda_2 - \lambda_1) \le b(g, \lambda_2 - \lambda_1) + b(g, \lambda_1 - \lambda_2). \end{aligned}$$

From this relations, taking into account (4.1), (4.2) and (4.4) and keeping in mind (4.9) we get (4.12), that concludes Step 2.

Step 3. Let $\tau \in [0, 1)$. Assume that for given $f, g \in X$ there exists a unique solution of Problem (4.9)–(4.10) with $r = \tau$, $(u, \lambda) \in X \times \Lambda$. Then, for given $f, g \in X$ there exists a unique solution (u, λ) of Problem (4.9)–(4.10) with $r \in [\tau, \tau + t_0] \subset [0, 1]$, where

$$0 < t_0 < \frac{\alpha\, m_a}{M_a\,(\alpha + m_a + 2\, M_a)} < 1. \tag{4.13}$$

Indeed, given $f, g \in X$, we define the mapping $\mathcal{T} : X \times \Lambda \to X \times \Lambda$ as follows

$$\mathcal{T}(w, \xi) := (u, \lambda)$$

if (u, λ) is the unique solution of the problem

$$\begin{aligned}
&a_\tau\,(u,v) + b(v,\lambda) = (F_s,\, v)_X, \qquad \forall v \in X,\\
&b(u,\mu-\lambda) \le b(g,\mu-\lambda), \qquad \forall \mu \in \Lambda,\\
&(F_s,\, v)_X = (f,\, v)_X - (s-\tau)\, c(w,\, v), \quad \tau \le s \le \tau + t_0 \le 1.
\end{aligned}$$

Taking into account (4.13), we deduce that $\mathcal{T}$ is a contraction. Using the Banach Fixed Point Theorem, we conclude that $\mathcal{T}$ has a unique fixed point. Let $(u^*,\, \lambda^*)$ be the unique fixed point of the operator $\mathcal{T}$. Using the definition of $\mathcal{T}$, we deduce

$$a_\tau(u^*,v) + b(v,\lambda^*) = (F_s,\, v)_X, \qquad \forall\, v \in X, \tag{4.14}$$
$$b(u^*,\mu-\lambda^*) \le b(g,\mu-\lambda), \qquad \forall\, \mu \in \Lambda, \tag{4.15}$$
$$(F_s,\, v)_X = (f,\, v)_X - (s-\tau)\, c(u^*,\, v), \quad \tau \le s \le \tau + t_0 \le 1. \tag{4.16}$$

Using now (4.14), (4.15) and (4.16) we deduce that $(u^*,\, \lambda^*)$ is a solution of Problem (4.9)–(4.10) with $r = s$, $s \in [\tau, \tau + t_0]$. In order to justify the uniqueness, let us assume that Problem (4.9)–(4.10) with $r = s$, $s \in [\tau, \tau + t_0]$ has two solutions $(u_1,\, \lambda_1)$, $(u_2,\, \lambda_2) \in X \times \Lambda$. Consequently, we can write

$$a_s(u_1-u_2, u_1-u_2) \le b(g, \lambda_2-\lambda_1) + b(g, \lambda_1-\lambda_2),$$

and from this, $a_s(u_1-u_2,\, u_1-u_2) \le 0$. Taking into account the X-ellipticity of a_s, we find $u_1 = u_2$. Moreover, using (4.4), we deduce that $\lambda_1 = \lambda_2$ that concludes Step 3.

Step 4. Using Step 3 a finite number of times, we deduce that Problem (4.9)–(4.10) admits a unique solution $(u, \lambda) \in X \times \Lambda$ for $r = 1$.

Step 5. In order to get (4.7), let us consider the data $f_1, g_1 \in X$ and $f_2, g_2 \in X$. Taking into account (4.5) and (4.6), we can write

$$a(u_1-u_2, u_1-u_2) \le (f_1-f_2, u_1-u_2)_X + b(g_1-g_2, \lambda_2-\lambda_1).$$

Moreover, keeping in mind (4.1)-(4.4), we deduce

$$m_a\|u_1-u_2\|_X^2 \le \|f_1-f_2\|_X\|u_1-u_2\|_X + M_b\|g_1-g_2\|_X\|\lambda_1-\lambda_2\|_Y, \tag{4.17}$$
$$\alpha\|\lambda_1-\lambda_2\|_Y \le M_a\|u_1-u_2\|_X + \|f_1-f_2\|_X. \tag{4.18}$$

Using (4.17), we can write

$$m_a\|u_1-u_2\|_X^2 \le \frac{\|f_1-f_2\|_X^2}{2k_1} + \frac{k_1\|u_1-u_2\|_X^2}{2} + \frac{M_b^2\|g_1-g_2\|_X^2}{2k_2} + \frac{k_2\|\lambda_1-\lambda_2\|_Y^2}{2},$$

where k_1, k_2 are strictly positive constants that will be chosen later. Combining the last inequality and (4.18) we deduce

$$\Big(m_a - \frac{k_1}{2} - \frac{k_2 M_a^2}{\alpha^2}\Big)\|u_1-u_2\|_X^2 \le \Big(\frac{1}{2k_1} + \frac{k_2}{\alpha^2}\Big)\|f_1-f_2\|_X^2 + \frac{M_b^2\|g_1-g_2\|_X^2}{2k_2}.$$

Choosing k_1, k_2 such that $(m_a - k_1/2 - k_2 M_a^2/\alpha^2) > 0$, we deduce that there exists $c^* = c^*(\alpha, m_a, M_a, M_b) > 0$ such that

$$\|u_1-u_2\|_X \le c^*(\|f_1-f_2\|_X + \|g_1-g_2\|_X). \tag{4.19}$$

Finally, combining (4.18) and (4.19) we deduce (4.12). □

Using Theorem 2 we can prove Theorem 1.

Proof. [Proof of Theorem 1]. Let us consider $X = \tilde{V}$, $Y = D$, and Λ given by (3.22). Obviously, Λ is a non-empty, closed, convex subset of H and $0_D \in \Lambda$. Using (3.8)–(3.10) we deduce that there exists $M_a = M_a(\mathcal{C}, \mathcal{E}, \boldsymbol{\beta}) > 0$ and $m_a = m_a(\mathcal{C}, \boldsymbol{\beta}) >0$ such that the bilinear form $a(\cdot,\cdot)$ satisfies

$$
\begin{aligned}
&|a(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}})| \le M_a \|\tilde{\boldsymbol{u}}\|_V \|\tilde{\boldsymbol{v}}\|_{\tilde{V}}, && \forall \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}} \in V,\\
&a(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}) \ge m_a \, \|\tilde{\boldsymbol{u}}\|^2_{\tilde{V}}, && \forall \tilde{\boldsymbol{u}} \in \tilde{V}.
\end{aligned}
$$

Taking into account (3.20) and (3.21) we deduce that the bilinear form $b(\cdot,\cdot)$ satisfies (4.3) and the inf-sup property (4.4). Consequently, Theorem 1 is a straightforward application of Theorem 2. □

5 Conclusions and Comments

We provided a mixed variational formulation for a frictionless unilateral contact problem involving electro-elastic materials. The main advantage of this type of formulation consists in the fact that it offers the possibility to use modern numerical techniques in order to write efficient algorithms for the approximation of the weak solution. This approach allows to approximate simultaneously the displacement field, the electric potential and the normal stress field.

A continuation of the study performed in this paper can be the writing of a discrete mortar formulation of Problem 1. Working on appropriate product spaces and following [9], mortar techniques with dual Lagrange multipliers can be applied in order to get an optimal a priori error estimate.

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