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ON THE STABILITY OF PIECEWISE POLYNOMIAL COLLOCATION METHODS FOR SOLVING WEAKLY SINGULAR INTEGRAL EQUATIONS OF THE SECOND KIND[1]

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Abstract. Piecewise polynomial collocation methods on special nonuniform grids are efficient methods for solving weakly singular Fredholm and Volterra integral equations but there is a widespread belief that those methods are numerically unstable in the case of large values of the nonuniformity parameter r. We show that this method by itself is stable and discuss some implementation problems that may lead to unstable behavior of numerical results.

Key words: weakly singular, Fredholm integral equation, Volterra integral equation, nonuniform grid, collocation method.

1 Introduction

We consider the linear weakly singular Volterra integral equations

$$y(t) = \int_0^t K(t,s)y(s)ds + f(t), \qquad t \in [0,T] \tag{1.1}$$

and Fredholm integral equations

$$y(t) = \int_0^T K(t,s)y(s)ds + f(t), \qquad t \in [0,T]. \tag{1.2}$$

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Note that Volterra equation can be considered to be a special case of a Fredholm equation with the kernel being equal to 0 above the diagonal $t = s$. Piecewise polynomial collocation methods using a graded grid of the form

$$t_n = T\left(\frac{n}{N}\right)^r, \ \ n = 0, 1, \dots, N$$

for Volterra equations and of the form (with even N)

$$t_n = \frac{T}{2}\left(\frac{2n}{N}\right)^r, \ n = 0, 1, \dots, \frac{N}{2}, \ \ t_{\frac{N}{2}+n} = T - t_{\frac{N}{2}-n}, \ \ n = 1, 2, \dots, \frac{N}{2} \quad (1.3)$$

for Fredholm equations with a suitable nonuniformity parameter $r \geq 1$ are among the most popular methods for solving this type of integral equations and have been studied quite extensively (see [5, 6, 7] for Fredholm equations and [1, 2, 3] for Volterra equations). Unfortunately many scientists and practitioners believe that those methods become unstable when the nonuniformity parameter r is large. We show that this method by itself is stable and discuss some implementation problems that may lead to unstable behavior of numerical results.

2 Assumptions and Smoothness Results

We assume that the kernel K belongs to the space $\mathcal{W}^{k,\nu}(D), \ k \in \mathbb{N}, \ \nu \in \mathbb{R}, \ \nu < 1$, where

$$D = \{(t, s) \in \mathbb{R}^2 : \ 0 \leq t \leq T, \ 0 \leq s \leq T, \ t \neq s\}.$$

The set $\mathcal{W}^{k,\nu}(D)$, with $k \in \mathbb{N}$, $\nu \in \mathbb{R}, \ \nu < 1$ consists of all k times continuously differentiable functions $K : D \to \mathbb{R}$ satisfying

$$\left|\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K(t, s)\right| \leq c \begin{cases} 1, & \text{if} \quad \nu + i < 0, \\ 1 + |\log|t - s||, & \text{if} \quad \nu + i = 0, \\ |t - s|^{-\nu - i}, & \text{if} \quad \nu + i > 0 \end{cases}$$

with a constant $c = c(K)$ for all $(t, s) \in D$ and all nonnegative integers i and j such that $i + j \leq k$.

In order to characterize the smoothness of the solutions of equation (1.2) we introduce the space $C_F^{k,\nu}[0, T]$, $k \in \mathbb{N}, \ \nu \in \mathbb{R}, \ \nu < 1$. It denotes the collection of all continuous functions $x : [0, T] \to \mathbb{R}$, which are k times continuously differentiable in $(0, T)$ such that the estimation

$$\left|x^{(i)}(t)\right| \leq c \begin{cases} 1, & \text{if} \quad i < 1 - \nu\,, \\ 1 + |\log \varrho(t)|, & \text{if} \quad i = 1 - \nu\,, \\ \varrho(t)^{1 - \nu - i}, & \text{if} \quad i > 1 - \nu \end{cases}$$

holds with $\varrho(t) = \min\{t, T - t\}\,, \ 0 < t < T\,$, and with a constant $c = c(x)$ for all $t \in (0, T)$ and $i = 1, \dots, k$.

Note that the derivatives staring from the order of $(1 - \lfloor \nu \rfloor)$ of the functions in $C_F^{k,\nu}[0, T]$ may be unbounded at both endpoints of the interval $[0, T]$. Now we are ready to state a smoothness result for solutions of the Fredholm equation.

Theorem 1. [6] *Let* $f \in C_F^{k,\nu}[0,T],\ K \in \mathcal{W}^{k,\nu}(D),\ k \in \mathbb{N},\ \nu \in \mathbb{R},\ \nu < 1$. *If the integral equation* (1.2)) *has a solution* $y \in L^\infty(0,T)$, *then* $y \in C_F^{k,\nu}[0,T]$.

It turns out that in the special case of Volterra equation (1.1), the solutions are smooth at $t = T$ if the data f is smooth at $t = T$. More precisely, let $C_V^{k,\nu}[0,T],\ k \in \mathbb{N},\ \nu \in \mathbb{R},\ \nu < 1$ be defined as the collection of all continuous functions $x : [0,T] \to \mathbb{R}$, which are k times continuously differentiable in $(0,T]$ and such that the estimation

$$\left|x^{(i)}(t)\right| \le c \begin{cases} 1, & \text{if} \quad i < 1-\nu, \\ 1 + |\log t|, & \text{if} \quad i = 1-\nu, \\ t^{1-\nu-i}, & \text{if} \quad i > 1-\nu \end{cases}$$

holds with a constant $c = c(x)$ for all $t \in (0,T]$ and $i = 0,1,\ldots,k$.

For the Volterra equation we have the following result

Theorem 2. [2] *Assume* $f \in C_V^{k,\nu}[0,T]$ *and* $K \in \mathcal{W}^{k,\nu}(D),\ k \in \mathbb{N},\ \nu \in \mathbb{R},\ \nu < 1$. *Then the integral equation* (1.1) *has a unique solution* $y \in C_V^{k,\nu}[0,T]$.

The smoothness results of Theorems 1 and 2 are sharp in the following sense: even if $f \in C^\infty[0,T]$ the solutions of the corresponding equations (1.2) and (1.1) have, in general, the singularities allowed by the spaces $C_F^{k,\nu}[0,T]$ and $C_V^{k,\nu}[0,T]$, respectively.

3 Grid and the Spline Spaces

For a given $N \in \mathbb{N}$ let

$$\Pi_N = \{t_0, t_1, \ldots, t_N : 0 = t_0 < t_1 < \ldots < t_N = T\}$$

be a partition (a mesh) of the interval $[0,T]$ (for ease of notation we suppress the index N in $t_n = t_n^{(N)}$ indicating the dependence of the grid points on N).

We look for approximate solutions to integral equations in the form of piecewise polynomial functions. Such functions are called polynomial splines.

Definition 1. Let k and d be given integers satisfying $-1 \le d \le k-1$. We call

$$\begin{aligned} S_k^{(d)}(\Pi_N) = \{w : w\big|_{(t_{n-1},t_n)} =: w_n \in \pi_k, \quad n = 1,\ldots,N; \\ w_n^{(i)}(t_n) = w_{n+1}^{(i)}(t_n) \quad 0 \le i \le d,\ n = 1,\ldots,N-1\} \end{aligned}$$

the space of (real) polynomial splines of degree k and of continuity class d. Here π_k denotes the set of polynomials of degree not exceeding k and $w\big|_{(t_{n-1},t_n)}$ is the restriction of $w : [0,T] \to \mathbb{R}$ to the subinterval (t_{n-1}, t_n).

4 Collocation Method

We define $m \geq 1$ interpolation points in every subinterval $[t_{n-1}, t_n]$ $(n = 1, \ldots, N)$ of the grid Π_N by

$$t_{nj} = t_{n-1} + \eta_j h_n, \quad j = 1, \ldots, m \quad (n = 1, \ldots, N),$$

where $h_n = t_n - t_{n-1}$ and $\eta_1, \ldots, \eta_m$ are some fixed parameters (called *collocation parameters*) which do not depend on n and N and satisfy

$$0 \leq \eta_1 < \ldots < \eta_m \leq 1.$$

We look for an approximate solution u to the solution y of equation (1.2) in $S^{(-1)}_{m-1}(\Pi_N)$, $m, N \in \mathbb{N}$. We determine $u = u^{(N)} \in S^{(-1)}_{m-1}(\Pi_N)$ by the collocation method from the following conditions:

$$u(t_{nj}) = \int_0^T K(t_{nj}, s)u(s)ds + f(t_{nj}), \quad j = 1, \ldots, m; \quad n = 1, \ldots, N.$$

If $\eta_1 = 0$, then by $u(t_{n1})$ we denote the right limit $\lim_{t \to t_{n-1}+0} u(t)$. Similarly, if $\eta_m = 1$, then $u(t_{nm})$ denotes the left limit $\lim_{t \to t_n - 0} u(t)$. Since the solutions of equation (1.2) are not smooth, it is necessary to use a suitably chosen nonuniform grid in order to achieve the optimal convergence rate.

5 "Unstable" Behavior of the Numerical Method

Consider an example of weakly singular Volterra integral equation

$$y(t) = \int_0^t |t-s|^{-\nu} y(s)ds + f(t), \quad t \in [0, T],$$

where $f(t)$ corresponds to the solution $y(t) = t^{1-\nu}$. In this case it is possible to find exact formulas (e.g. by using tables of integrals) for the integrals needed for forming the system of collocation equations. In the case $\nu = 0.5$, $m = 3$, and collocation parameters $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{2}$, $\eta_3 = \frac{3}{4}$ the theoretical convergence rate in the supremum norm is (see [2])

$$\|y - u\|_\infty \leq c \begin{cases} N^{-r(1-\nu)}, & \text{if } 1 \leq r < \dfrac{m}{1-\nu}, \\ N^{-m}(1 + \log N), & \text{if } r = \dfrac{m}{1-\nu} = 1, \\ N^{-m}, & \text{if } r > \dfrac{m}{1-\nu} \text{ or } r = \dfrac{m}{1-\nu} > 1, \end{cases}$$

but the straightforward numerical implementation gives the results presented in Table 1. In the table $\varepsilon_N = \|u^{(N)} - y\|_\infty$ denotes the actual error, in the columns with the labels ρ are the quotients of the errors for the values of N that are the consecutive powers of 2 and the number in the parenthesis

Table 1. Numerical results, "exact" system integrals.

$\nu = 0.5$	$r = 1$		$r = 2$		$r = 4$		$r = 8$	
N	ε_N	$\varrho(1.4)$	ε_N	$\varrho(2.0)$	ε_N	$\varrho(4.0)$	ε_N	$\varrho(8.0)$
4	1.7E-2	1.2	1.1E-2	1.6	6.3E-3	2.8	1.4E-2	1.9
8	1.4E-2	1.2	6.0E-3	1.8	1.9E-3	3.4	3.1E-3	4.5
16	1.1E-2	1.3	3.2E-3	1.9	4.9E-4	3.8	1.4E-2	0.2
32	8.1E-3	1.3	1.7E-3	1.9	1.2E-4	4.0	7.7E+1	0.0
64	6.0E-3	1.3	8.4E-4	2.0	3.1E-5	4.0	6.8E+5	0.0
128	4.4E-3	1.4	4.2E-4	2.0	1.1E-3	0.0	7.4E+7	0.0
256	3.2E-3	1.4	2.1E-4	2.0	6.7E-2	0.0	1.5E+8	0.5
512	2.3E-3	1.4	1.1E-4	2.0	7.8E+0	0.0	3.6E+8	0.4
1024	1.7E-3	1.4	5.3E-5	2.0	4.8E+2	0.0	1.9E+8	1.9

behind ρ shows what the quotient of the errors should be if the theoretical error estimates were exact. As we see, in the case of relatively small values of r the computational results are in a good agreement with the theoretical error estimates but for larger values of the nonuniformity parameter r the numerical results show a clear unstable behavior. In the next section we analyze the possible causes of numerical instability and identify the reason of the clearly unsatisfactory results obtained for $r \geq 4$ in the case of the test equation.

6 Numerical Stability of the Piecewise Polynomial Collocation Method

Let us first recall some general results about stability of solution methods for linear equations following the framework presented in [4, Chapter 1.3]. Let X be a Banach space and let $A \in \mathcal{L}(X)$ be an invertible linear operator. Consider the equation

$$Ax = f, \qquad x, f \in X.$$

When constructing a numerical method for the equation one usually introduces a finite dimensional subspace X_n of X, replaces A with an approximation A_n that maps X_n to X_n and replaces the right hand side f with an approximation from X_n:

$$A_n x_n = f_n, \qquad x_n, f_n \in X_n.$$

This defines a numerical method but for actual implementations it is also necessary to reduce it to a matrix equation. For this purposes one usually introduces a basis $\{e_i\}_{i=1}^n$ in X_n and reduces the previous operator equation to the matrix equation for the coefficients of the approximate solution x_n relative to the chosen basis. By defining

$$R_n : \mathbb{R}^n \to X_n \qquad R_n c = \sum_{i=1}^n c_i e_i$$

and

$$L_n = R_n^{-1} : X_n \to \mathbb{R}^n \qquad [L_n(\sum_{i=1}^n c_i\, e_i,)]_j = c_j,$$

we can write the resulting linear matrix equation as

$$Mc = b,$$

where $M = L_n A_n R_n$ and $b = L_n f_n$; our approximate solution to the original equation is then $x_n = R_n c$.

It is usually not possible to compute the values of the coefficient matrix M and the vector b exactly, so we actually solve the matrix equation

$$(M + \delta M)(c + \delta c) = b + \delta b$$

and the actual error of the numerical solution is $\|x - x_n - R_n \delta c\|$ which we can split into two parts:

$$\|x - x_n - R_n \delta c\| \leq \|x - x_n\| + \|R_n \delta c\|.$$

The first part depends on how well our method approximates the original equation and the second part depends on the stability of the resulting matrix equation. Using the condition number of M defined by

$$\mathrm{cond}(M) = \|M\| \|M^{-1}\|$$

we have

$$\frac{\|\delta c\|}{\|c\|} \leq \frac{\mathrm{cond}(M)}{1 - \|\delta M\| \|M^{-1}\|} \left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta M\|}{\|M\|} \right),$$

hence the uniform (with respect to n) boundedness of R_n and $\mathrm{cond}(M)$ together with good approximation properties of the numerical method imply the stability of the actual numerical implementation, provided the errors in computing the coefficients of the linear system and the right hand side remain small; instability of the numerical method may be caused either by the bad approximation properties of the method or by a bad choice of the basis in X_n leading to unbounded condition number of the matrix M.

Let us analyze the piecewise polynomial collocation method for solving weakly singular integral equations according to the presented scheme. Denote by T the (Fredholm or Volterra) integral operator and by P_N the piecewise polynomial interpolation operator defined on the grid Π_N, then the collocation method for the original equation

$$y = Ty + f$$

can be written as

$$u - P_N T u = P_N f.$$

In the proofs of the convergence theorems for Volterra and Fredholm equations it is shown that the method has good approximation properties and that $(I - P_N T)$ converges to $(I - T)$ in $\mathcal{L}(L^\infty)$ (see [2] and [6], respectively). Hence (assuming additionally that the kernel of $I - T$ contains only the zero element in the case of Fredholm equation) we have that the operator $A_N = I - P_N T$ has a finite condition number in the space $L^\infty(0, T)$ that converges to the condition number of the operator $I - T$, thus it has a finite condition that is independent of r also in the space of piecewise polynomial functions defined on Π_N. Now, if we use for the basis functions in $S_{m-1}^{(-1)}$ the piecewise Lagrangian polynomial functions $\phi_{ij} \in S_{m-1}^{(-1)}$ defined by

$$\phi_{ij}(t) = \begin{cases} \phi_j\Big(\dfrac{t - t_{i-1}}{t_i - t_{i-1}}\Big), & \text{if } \ t \in [t_{i-1}, t_i], \\ 0, & \text{otherwise}, \end{cases}$$

where the functions ϕ_j are the Lagrangian polynomials of degree $m-1$ corresponding to the collocation parameters $\eta_k,\ k = 1, \ldots, m$ and use the maximum norm in $\mathbb{R}^n$, then clearly we have

$$\|L_n w\|_\infty \le \|w\|_\infty \quad \forall w \in S_{m-1}^{(-1)}$$

and

$$\|R_n c\|_\infty \le \lambda_m |c\|_\infty,\ \ \lambda_m = \max_{0 \le t \le 1} \sum_{k=1}^{m} |\phi_k(t)| = \|P_m\|_{C[0,1] \to L^\infty(0,1)},$$

where $n = Nm$ is the dimension of the space $S_{m-1}^{(-1)}$, and P_m is the interpolation operator corresponding to the functions $\phi_k,\ k = 1 \ldots, m$. As $L_n^{-1} = R_n$ we now have

$$\text{cond}(M) = \|L_n A_N R_n\| \, \|R_n^{-1} A_N^{-1} L_n^{-1}\| \le \|R_n\|_\infty^2 \text{cond}(A_N) \le \text{const}.$$

Therefore we may conclude that in the case of Lagrangian basis functions the piecewise polynomial collocation method is stable provided the coefficient matrix is computed with reasonable accuracy. Thus the "unstable" numerical results obtained previously were just the result of careless application of Newton-Leibniz rule for computing system integrals that introduced large round-off errors.

On the basis of the analysis we may conclude that the piecewise polynomial collocation method using special nonuniform grids is stable for solving weakly singular integral equations, provided the system integrals are computed with sufficient accuracy. There are at least three ways to achieve this:

- by using high precision arithmetics in computing the formulas of system integrals;
- by rewriting the exact formulas in a form that is not sensitive to round-off errors;
- by computing the system integrals numerically with sufficient accuracy.

In order to demonstrate the accuracy of the conclusions we present the numerical results for the same method and test equation in the case when the system integrals are computed numerically. We computed the system integrals with relative error less than 10^{-10} using a quadrature formula on a nonuniform grid together with Runge's error estimate. This made the solution procedure about 10 times slower than in the case of the "exact" system integrals (independently of the value of N). As we see in Table 2, the results are now in complete agreement with the theoretical error estimates.

Table 2. Numerical results, numerically computed system integrals.

	$r=4$		$r=8$		$r=16$		$r=32$	
N	ε_N	$\varrho(4.0)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$	ε_N	$\varrho(8.0)$
4	6.3E-3	2.8	1.4E-2	1.9	2.6E-2	1.1	2.8E-2	1.0
8	1.9E-3	3.4	3.1E-3	4.5	1.3E-2	2.0	2.5E-2	1.1
16	4.9E-4	3.8	4.6E-4	6.8	2.9E-3	4.4	1.2E-2	2.1
32	1.2E-4	4.0	6.0E-5	7.7	4.4E-4	6.5	2.8E-3	4.3
64	3.1E-5	4.0	7.5E-6	8.0	5.9E-5	7.5	4.3E-4	6.4
128	7.7E-6	4.0	9.3E-7	8.1	7.4E-6	7.9	5.8E-5	7.5
256	1.9E-6	4.0	1.1E-7	8.1	9.2E-7	8.0	7.4E-6	7.9
512	4.8E-7	4.0	1.4E-8	8.1	1.1E-7	8.1	9.2E-7	8.0
1024	1.2E-7	4.0	1.8E-9	8.0	1.4E-8	8.1	1.1E-7	8.0

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