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ON FUČIK SPECTRA FOR THIRD ORDER EQUATIONS

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Abstract. We construct the Fučik spectrum for some third order nonlinear boundary value problems. This spectrum differs essentially from the known Fučik spectra.
Key words: Fučik problem, Fučik spectrum

1. Introduction

In this paper we study the Fučik spectra for third order equations with piecewise linear right sides. Investigations of the Fučik spectra have started fifty years ago [3]. A number of authors have studied the specific cases. Let us mention the cases of the Dirichlet [3] and the Sturm-Liouville [5] boundary conditions. There are some papers on higher order equations. Habets and Gaudenzi have studied the third order problem with the boundary conditions $x(0) = x'(0) = 0 = x(1)$ in [1], where many useful references on the subject can be found. The Fučik spectrum for the fourth order equations was considered by Kreiči [2] and Pope [4].

The paper is organized as follows. In Section 2 we present results on the Fučik spectrum for the third order problem with the boundary conditions $x(a) = x'(a) = 0 = x(b)$ and compare them with the results for the boundary conditions $x(a) = x'(a) = 0 = x'(b)$. In the proof we reduce the third order problem to the second order problem with the boundary conditions including a nonlocal (integral) condition. We construct the Fučik spectra for these problems. These are the main results of the work. A connection between the spectra are discussed in Section 3.

2. The Fučik Spectra for Some Third Order Boundary Value Problems

Consider a boundary value problem

$$\begin{cases} x''' = -\mu^2 x'^{+} + \lambda^2 x'^{-}, \quad \mu, \quad \lambda > 0, \\ x(a) = 0, \quad x'(a) = 0, \quad x(b) = 0, \end{cases} \tag{2.1}$$

where we use notation

$$x'^{+} = \max\{x', 0\}, \quad x'^{-} = \max\{-x', 0\}.$$

Definition 1. The Fučik spectrum is a set of points (λ, μ) such that problem (2.1) has nontrivial solutions.

The first result describes a decomposition of the spectrum into branches F_i^+ and F_i^- $(i = 0, 1, 2, \ldots)$.

Proposition 1. *The Fučik spectrum consists of the set of curves*

$$F_i^+ = \{(\lambda, \mu) : \ x''(a) > 0, \text{ the derivative } x'(t) \textit{ of a nontrivial solution of the problem has exactly } i \textit{ zeroes in } (a, b)\},$$

$$F_i^- = \{(\lambda, \mu) : \ x''(a) < 0, \textit{ the derivative } x'(t) \textit{ of a nontrivial solution of the problem has exactly } i \textit{ zeroes in } (a, b)\}.$$

Now we formulate the main result of this work.

Theorem 1. *The Fučik spectrum for the problem* (2.1) *consists of the branches given by*

$$F_{2i-1}^+ = \Big\{(\lambda, \mu) : \ \frac{2i\lambda}{\mu} - \frac{(2i-1)\mu}{\lambda} - \frac{\mu\cos(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i)}{\lambda} = 0,$$
$$\frac{i\pi}{\mu} + \frac{(i-1)\pi}{\lambda} < b - a, \ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > b - a\Big\},$$

$$F_{2i}^+ = \Big\{(\lambda, \mu) : \ \frac{(2i+1)\lambda}{\mu} - \frac{2i\mu}{\lambda} - \frac{\lambda\cos(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i)}{\mu} = 0,$$
$$\frac{i\pi}{\mu} + \frac{i\pi}{\lambda} < b - a, \ \frac{(i+1)\pi}{\mu} + \frac{i\pi}{\lambda} > b - a\Big\},$$

$$F_{2i-1}^- = \Big\{(\lambda, \mu) : \ \frac{2i\mu}{\lambda} - \frac{(2i-1)\lambda}{\mu} - \frac{\lambda\cos(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i)}{\mu} = 0,$$
$$\frac{(i-1)\pi}{\mu} + \frac{i\pi}{\lambda} < b - a, \ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > b - a\Big\},$$

$$F_{2i}^- = \Big\{(\lambda,\mu):\ \frac{(2i+1)\mu}{\lambda} - \frac{2i\lambda}{\mu} - \frac{\mu\cos(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i)}{\lambda} = 0,$$
$$\frac{i\pi}{\mu} + \frac{i\pi}{\lambda} < b-a,\ \frac{i\pi}{\mu} + \frac{(i+1)\pi}{\lambda} > b-a\Big\},$$

where $i = 1, 2, \ldots$.

Proof. Let us consider problem (2.1). We introduce the notation $x' = y$, then problem (2.1) reduces to

$$y'' = -\mu^2 y^+ + \lambda^2 y^-,\ \ \mu,\lambda > 0, \tag{2.2}$$
$$y^+ = \max\{y, 0\}, \quad y^- = \max\{-y, 0\},$$

with the boundary conditions

$$y(a) = 0, \quad \int_a^b y(s)ds = 0. \tag{2.3}$$

Let us set $x(t) = \int_a^t y(s)ds$, then $x'(t) = y(t)$ and equation (2.2) reduces to the equation (2.1). It follows from conditions (2.3) that

$$x'(a) = y(a) = 0, \quad x(a) = \int_a^a y(s)ds = 0, \quad x(b) = \int_a^b y(s)ds = 0.$$

That is why problems (2.1) and (2.2), (2.3) are equivalent.

In the following we consider problem (2.2), (2.3). It is clear that $y(t)$ must have zeroes in (a, b). That is why $F_0^\pm = \emptyset$. We will prove the theorem for the case of F_1^+. Suppose that $(\lambda,\mu) \in F_1^+$ and let $y(t)$ be a respective nontrivial solution of problem (2.2), (2.3). The solution has only one zero in (a, b) and $y'(a) > 0$. Let us denote this zero by τ.

Consider a solution of problem (2.2), (2.3) in the interval (a,τ) and in the interval (τ, b). We obtain that problem (2.2), (2.3) in these intervals reduces to the linear eigenvalue problems. So in the interval (a,τ) we have the problem

$$\begin{cases} y'' = -\mu^2 y, \\ y(a) = 0,\ \ y(\tau) = 0, \end{cases}$$

but in the interval (τ, b) we have the problem $y'' = -\mu^2 y$ with boundary condition $y(\tau) = 0$. In view of (2.3) solution $y(t)$ must satisfy the condition

$$\int_a^\tau y(s)\,ds = \Big|\int_\tau^b y(s)\,ds\Big|. \tag{2.4}$$

Since $y(t) = A\sin(\mu t - \mu a)$ ($A > 0$) and $y(\tau) = 0$ we obtain $\tau = \dfrac{\pi}{\mu} + a$. In view of this equality it is easy to get that

$$\int_a^\tau y(s)\,ds = \frac{A}{\mu}\big(1 - \cos\mu(\tau - a)\big) = \frac{2A}{\mu}.$$

We have also

$$y'(\frac{\pi}{\mu} + a) = -\mu A. \tag{2.5}$$

Now we consider a solution of problem (2.2), (2.3) in $[\tau, b]$. Since a general solution is given by $y(t) = -B\sin(\lambda t - \lambda\frac{\pi}{\mu} - \lambda a)$ $(B > 0)$, we obtain

$$\Big|\int_\tau^b y(s)ds\Big| = \frac{B}{\lambda}\big(1 - \cos(\lambda b - \lambda\frac{\pi}{\mu} - \lambda a)\big).$$

We have also that

$$y'(\frac{\pi}{\mu} + a) = -\lambda B. \tag{2.6}$$

It follows from (2.5) and (2.6) that $A = \dfrac{\lambda B}{\mu}$. In view of the last equality and (2.4) we obtain that

$$\frac{2\lambda B}{\mu^2} = \frac{B}{\lambda}\big(1 - \cos(\lambda b - \lambda\frac{\pi}{\mu} - \lambda a)\big).$$

Dividing it by B and multiplying by μ, we obtain that

$$\frac{2\lambda}{\mu} - \frac{\mu}{\lambda} + \frac{\mu\cos(\lambda(b-a) - \frac{\lambda\pi}{\mu})}{\lambda} = 0. \tag{2.7}$$

Considering the solution of problem (2.2), (2.3) it is easy to prove that $a < \dfrac{\pi}{\mu} < b < \dfrac{\pi}{\mu} + \dfrac{\pi}{\lambda}$. This result and (2.7) prove the theorem for the case of F_1^+. The proof for other branches is analogous. ■

Corollary 1. The spectrum of problem (2.2), (2.3) is given by formulas from Theorem 1.

Now let us consider the spectrum of the problem

$$\begin{cases} x''' = -\mu^2 x'^+ + \lambda^2 x'^-, \quad \mu, \quad \lambda > 0, \\ x(a) = 0, \quad x'(a) = 0, \quad x'(b) = 0. \end{cases} \tag{2.8}$$

A decomposition of the Fučik spectrum for problem (2.8) into branches F_i^+ and F_i^- $(i = 1, 2, \ldots)$ is the same as that for problem (2.1).

Theorem 2. *The Fučik spectrum for* (2.8) *consists of the following branches:*

$$F_0^+ = \Big\{(\lambda, \frac{\pi}{b-a})\Big\}, \quad F_0^- = \Big\{(\frac{\pi}{b-a}, \mu)\Big\},$$

$$F_{2i-1}^+ = \Big\{(\lambda,\mu): \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} = b-a\Big\},\ F_{2i}^+ = \Big\{(\lambda,\mu): \frac{(i+1)\pi}{\mu} + \frac{i\pi}{\lambda} = b-a\Big\},$$

$$F_{2i-1}^- = \Big\{(\lambda,\mu): \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} = b-a\Big\},\ F_{2i}^- = \Big\{(\lambda,\mu): \frac{i\pi}{\mu} + \frac{(i+1)\pi}{\lambda} = b-a\Big\},$$

where $i = 1, 2, \ldots$.

Proof. Consider problem (2.8). We introduce the following notation $x' = y$. Then problem (2.8) reduces to the Fučik problem

$$\begin{cases} y'' = -\mu^2 y^+ + \lambda^2 y^-, & \mu, \lambda > 0, \\ y(a) = 0, \quad y(b) = 0. \end{cases} \tag{2.9}$$

Set $x(t) = \int_a^t y(s)\,ds$, then $x'(t) = y(t)$ and equation (2.9) reduces to equation (2.1). In view of the conditions of problem (2.9), we obtain that

$$x'(a) = 0,\ \ y(a) = 0,\ \ x'(b) = 0,\ \ y(b) = 0,\ \ x(a) = \int_a^a y(s)\,ds = 0.$$

Notice that problem (2.9) is the classical Fučik problem, which was investigated in [3]. The proof of this theorem is given in [3]. ■

Corollary 2. The spectrum of the problem (2.9) is given by formulas from Theorem 2.

A visualization of the spectrum of problems (2.1) and (2.8) in the case of $a = 0$, $b = 1$ is given in Fig. 1 and Fig. 2.

At the end of this section we would like to give some properties of the spectra of problems (2.1) and (2.8):

- branches of spectrum for problem (2.1) are finite, while branches of spectrum for problem (2.8) are infinite;
- all positive branches F_i^+ constitute a continuous curve, which is located above the bisectrix, similarly all negative branches F_i^- constitute a continuous curve, which is located below the bisectrix for problem (2.1);
- the curves $F_{2n-1}^{\pm}$ and $F_{2n}^{\pm}$ have a common point which is eigenvalue of the corresponding linear problem, the curves $F_{2n}^{\pm}$ and $F_{2n+1}^{\pm}$ have a common point which is not eigenvalue of the respective linear problem for the associated problem (2.1).

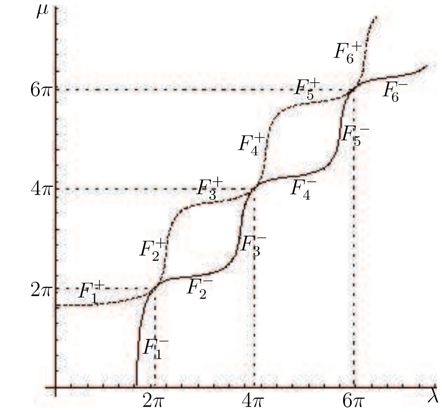

Figure 1. The Fučik spectrum for problem (2.1).

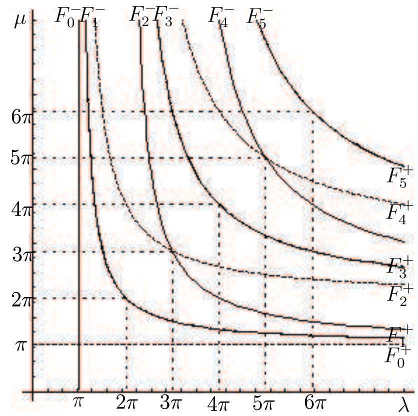

Figure 2. The Fučik spectrum for problem (2.8).

3. Connection Between the Spectra

Consider the boundary value problem

$$\begin{cases} x''' = -\mu^2 x'^{+} + \lambda^2 x'^{-}, \quad \mu, \quad \lambda > 0, \\ x(a) = x'(a) = 0, \quad \alpha x(b) + (1-\alpha)x'(b) = 0, \quad \alpha \in [0,1]. \end{cases} \tag{3.1}$$

Theorem 3. *The Fučik spectrum for problem* (3.1) *consists of the branches given by*

$$F^{+}_{2i-1} = \Big\{(\lambda,\mu):\ \frac{2i\lambda}{\mu}\alpha - \frac{(2i-1)\mu}{\lambda}\alpha - \frac{\mu\alpha\cos\Big(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\Big)}{\lambda}$$
$$+ \mu\sin\Big(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\Big) - \alpha\mu\sin\Big(\lambda(b-a) - \frac{\lambda\pi i}{\mu} + \pi i\Big) = 0,$$
$$\frac{i\pi}{\mu} + \frac{(i-1)\pi}{\lambda} < b-a,\ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > b-a\Big\},$$

$$F^{+}_{2i} = \Big\{(\lambda,\mu):\ \frac{(2i+1)\lambda}{\mu}\alpha - \frac{2i\mu}{\lambda}\alpha - \frac{\lambda\alpha\cos\Big(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i\Big)}{\mu}$$
$$+ \lambda\sin\Big(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i\Big) - \alpha\lambda\sin\Big(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i\Big) = 0,$$
$$\frac{i\pi}{\mu} + \frac{i\pi}{\lambda} < b-a,\ \frac{(i+1)\pi}{\mu} + \frac{i\pi}{\lambda} > b-a\Big\},$$

$$F^{-}_{2i-1} = \Big\{(\lambda,\mu):\ \frac{2i\mu}{\lambda}\alpha - \frac{(2i-1)\lambda}{\mu}\alpha - \frac{\lambda\alpha\cos\Big(\mu(b-a) - \frac{\mu\pi i}{\lambda} + \pi i\Big)}{\mu}$$

$$+\lambda\sin\Big(\mu(b-a)-\frac{\mu\pi i}{\lambda}+\pi i\Big)-\alpha\lambda\sin\Big(\mu(b-a)-\frac{\mu\pi i}{\lambda}+\pi i\Big)=0,$$
$$\frac{(i-1)\pi}{\mu}+\frac{i\pi}{\lambda}<b-a,\ \frac{i\pi}{\mu}+\frac{i\pi}{\lambda}>b-a\Big\},$$

$$F_{2i}^{-}=\Big\{(\lambda,\mu):\ \frac{(2i+1)\mu}{\lambda}\alpha-\frac{2i\lambda}{\mu}\alpha--\frac{\mu\alpha\cos\Big(\lambda(b-a)-\frac{\lambda\pi i}{\mu}+\pi i\Big)}{\lambda}$$
$$+\mu\sin\Big(\lambda(b-a)-\frac{\lambda\pi i}{\mu}+\pi i\Big)-\alpha\mu\sin\Big(\lambda(b-a)-\frac{\lambda\pi i}{\mu}+\pi i\Big)=0,$$
$$\frac{i\pi}{\mu}+\frac{i\pi}{\lambda}<b-a,\ \frac{i\pi}{\mu}+\frac{(i+1)\pi}{\lambda}>b-a\Big\}.$$

Remark 1. If $\alpha=0$ we obtain problem (2.1). In case of $\alpha=1$ we have problem (2.8).

The branches $F_1^{\pm}$ to $F_5^{\pm}$ of the spectrum for problem (3.1) are presented in Fig. 3 for several values of α in the case of $a=0$, $b=1$.

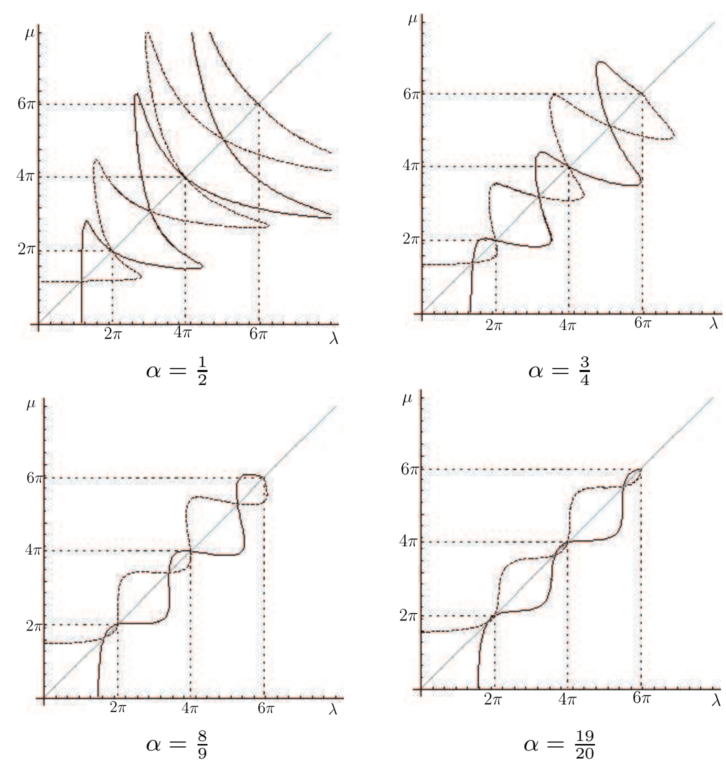

Figure 3. The Fučik spectrum for problem (3.1) for some values of α.

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