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GRID APPROXIMATION OF SINGULARLY PERTURBED PARABOLIC REACTION-DIFFUSION EQUATIONS WITH PIECEWISE SMOOTH INITIAL-BOUNDARY CONDITIONS [1]

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Dedicated to Pieter W. Hemker on the occasion of his 65th birthday

Abstract. A Dirichlet problem is considered for a singularly perturbed parabolic reaction–diffusion equation with piecewise smooth initial-boundary conditions on a rectangular domain. The higher-order derivative in the equation is multiplied by a parameter ε^2; $\varepsilon \in (0, 1]$. For small values of ε, a boundary and an interior layer arises, respectively, in a neighbourhood of the lateral part of the boundary and in a neighbourhood of the characteristic of the reduced equation passing through the point of nonsmoothness of the initial function. Using the method of special grids condensing either in a neighbourhood of the boundary layer or in neighbourhoods of the boundary and interior layers, special ε–uniformly convergent difference schemes are constructed and investigated. It is shown that the convergence rate of the schemes *crucially* depends on the type of *nonsmoothness* in the initial–boundary conditions.

Key words: singularly perturbed boundary value problem, piecewise smooth initial–boundary conditions, parabolic reaction–diffusion equation, finite difference approximation, ε-uniform convergence, compatibility conditions, special grids

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1. Introduction

Difficulties in solving boundary value problems for singularly perturbed equations (equations involving a small parameter ε multiplying the higher-order derivatives) with sufficiently smooth data are well known (see, e.g., [1, 2, 4, 8, 9, 15] and also their bibliographies). For problems of this type, special numerical methods are required that allow us to approximate the solutions with an error bound that is independent of the parameter ε, i.e., ε–uniformly convergent methods. Usually, when constructing and investigating ε–uniformly convergent schemes, the data of boundary value problems are assumed to be sufficiently smooth and satisfying additional compatibility conditions [6] ensuring the smoothness of solutions for the studied problems. Difficulties in approximating the solution increase when the initial or boundary conditions are piecewise smooth. Boundary value problems for singularly perturbed parabolic reaction–diffusion problems with a strong singularity — a discontinuous initial condition — were considered in [3, 5, 11, 12, 16, 17]. In these problems, besides the fitted mesh method (meshes condensing in a neighbourhood of the boundary layers), a specific technique was used so as the fitted operator method [3, 11, 12], [16], or the method of the additive splitting of singularities (in a neighbourhood of the points of discontinuities in the initial function) [5, 17].

Special schemes for a singularly perturbed parabolic convection–diffusion equations with only a single weak singularity — a piecewise smooth initial condition — were considered in [7, 18, 19]. In these papers, it is assumed that conditions on the lateral boundary of the domain are sufficiently smooth, moreover, compatibility conditions at the corner points are fulfilled that ensure the solution belonging to $C^{4+\alpha,2+\alpha/2}$ in a neighbourhood of the lateral boundary of the domain. However, investigations of special difference schemes for singularly perturbed parabolic equations with several weak singularities such as piecewise smooth boundary and initial conditions and/or the absence of compatibility conditions as well (in particular, for reaction-diffusion equations) are not known in the literature. Thus, the development of ε–uniformly convergent difference schemes for a class of singularly perturbed problems with additional singularities marked above is an actual problem.

In the present paper, we consider a Dirichlet problem on a rectangular domain for a singularly perturbed parabolic reaction-diffusion equation with a small parameter ε^2 multiplying the higher-order derivative. The first-order derivatives with respect to x of the initial function and/or with respect to t of the boundary function have jump discontinuities; compatibility conditions at the corner points (except the continuity condition) are not assumed. When the parameter ε tends to zero, boundary and interior parabolic layers arise. Here, ε-uniform convergence of special difference schemes on piecewise uniform condensing meshes is investigated depending on nonsmoothness of various types in the initial–boundary conditions.

It is shown that in the case of a *standard problem* — a problem with sufficiently smooth data satisfying the compatibility condition – the known classical finite difference scheme on the special piecewise uniform mesh condensing in the *boundary layer* (we call it *the basic scheme*) converges ε-uniformly at the rate $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-1})$, where $N+1$ and N_0+1 are the numbers of nodes in the meshes with respect to x and to t, respectively. The appearance of nonsmoothness of such types as the *discontinuity of the first-order t-derivative of the boundary function* and/or the *absence of compatibility conditions* only weakly influences the rate of ε–uniform convergence. In this case, the basic scheme converges at the rate $\mathcal{O}(N^{-2}\ln^3 N + N_0^{-1}\ln N_0)$.

Nonsmoothness of such type as the *discontinuity of the first-order derivative with respect to x of the initial function* essentially influences the ε–uniform convergence rate. In this case, the basic scheme converges only at the rate $\mathcal{O}(N^{-1} + N_0^{-1/2})$; a significant decrease in the order of the ε–uniform convergence rate takes place. For this problem, an *improved scheme* is constructed — the scheme on the meshes condensing in both boundary and interior layers – that allows us to obtain discrete solutions convergent *conditionally ε–uniformly* at the rate $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-1}\ln N_0)$ under the condition that the parameter ε satisfies the additional condition $\varepsilon = \mathcal{O}(N^{-1} + N_0^{-1/2})$. On the other hand, *the unconditional ε–uniform* convergence rate of the improved scheme is $\mathcal{O}(N^{-1} + N_0^{-1/2})$, i.e., the same as it is for the basic scheme.

2. Problem Formulation and Aim of Research

2.1. Set

$$\overline{G} = G \cup S, \quad G = D \times (0,\, T], \quad D = \{x:\ x \in (-d,\, d)\}. \tag{2.1}$$

On $\overline{G}$ we consider the Dirichlet problem for the singularly perturbed parabolic reaction-diffusion equation [2]

$$\begin{cases} L_{(2.2)}\, u(x,t) = f(x,t), & (x,t) \in G, \\ u(x,t) = \varphi(x,t), & (x,t) \in S. \end{cases} \tag{2.2}$$

Here

$$L_{(2.2)} \equiv \varepsilon^2\, a(x,t)\, \frac{\partial^2}{\partial x^2} - c(x,t) - p(x,t)\, \frac{\partial}{\partial t},$$

the coefficients $a(x,t)$, $c(x,t)$, $p(x,t)$ and the right–hand side $f(x,t)$ are sufficiently smooth on the set $\overline{G}$, moreover,

$$a(x,t) \ge a_0, \quad c(x,t) \ge 0, \quad p(x,t) \ge p_0, \quad (x,t) \in \overline{G}; \quad a_0,\, p_0 > 0;$$

and the parameter ε takes arbitrary values in the half-open interval $(0,\, 1]$.

The initial–boundary function $\varphi(x,t)$ is continuous on S and is piecewise smooth on the sets S_0 and $\overline{S}^{L}$. Here $S = S_0 \bigcup S^L$, $S_0 = \overline{S}_0$ and S^L are the lower and lateral parts of the boundary S, $\ S^L = \Gamma \times (0,\, T]$, $\ \Gamma = \overline{D} \setminus D$.

[2] Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, grids) are defined in formula $(j.k)$.

The first-order derivative of $\varphi(x,t)$ with respect to t has a jump discontinuity on the set $\overline{S}^{\,L}$; and the first-order derivative of $\varphi(x,t)$ with respect to x has a jump discontinuity on the set S_0, namely, at the point $(0,\,0)$. The function $\varphi(x,t)$ is assumed to be sufficiently smooth on the closure of those parts to the boundary S, on which the first-order derivatives are continuous. In particular, the function $\varphi(x,t)$ is sufficiently smooth on the sets S_0^+ and S_0^-, $\varphi(\cdot,\,0) \subset C(S_0)\bigcap\{C^k(S_0^-)\bigcup C^k(S_0^+)\}$, $k \geq 1$, where $S_0 = S_0^-\bigcup S_0^+$, $S_0^- = S_0\bigcap\{x \leq 0\}$, $S_0^+ = S_0\bigcap\{x \geq 0\}$.

The fulfillment of compatibility conditions [6] on the set of the corner points $S_* = S_0\bigcap\overline{S}^{\,L}$ are not assumed.

By a solution of problem (2.2), we mean a function $u \in C(\overline{G}) \cap C^{2,1}(G)$ that satisfies the differential equation on G and the boundary condition on S.

For fixed values of the parameter ε, the derivative $(\partial/\partial x)u(x,t)$ is continuous on $\overline{G}^{\,*}$, where $\overline{G}^{\,*} = \overline{G}\setminus(0,\,0)$, it is also bounded on $\overline{G}^{\,*}$, and is discontinuous at the point $(0,0)$. The derivatives $(\partial^2/\partial x^2)\,u(x,t)$, $(\partial/\partial t)\,u(x,t)$ have a jump discontinuity at the point $(0,\,0)$ and on the set S_*, and also on the set of points in $\overline{S}^{\,L}$, where the first-order derivative of the function $\varphi(x,t)$ with respect to t has a discontinuity.

Let us discuss the behaviour of the solution for small values of the parameter ε. Let $S^{\gamma} = \{(x,t):\ x = \gamma(t) \equiv 0,\ t \in [0,\,T]\}$ be the characteristic of the reduced equation that passes through the point $(0,\,0)$. When the parameter ε tends to zero, a boundary and an interior layer with the characteristic length scale ε appears in a neighbourhood of the sets $S^{\,L}$ and S^{γ} respectively, moreover, the interior layer is weak; see bounds (3.10) in Section 3. It is known that solutions of classical finite difference schemes do not converge ε-uniformly (see, e.g., [2, 5]) even in the case of singularly perturbed problems with sufficiently smooth data.

2.2. On the set $\overline{G}$, we consider also the boundary value problem for the singularly perturbed equation with constant coefficients

$$\begin{cases} L_{(2.3)}\,u(x,t) = f(x,t), & (x,t) \in G, \\ u(x,t) = \varphi(x,t), & (x,t) \in S. \end{cases} \tag{2.3}$$

Here

$$L_{(2.3)} \equiv \varepsilon^2\,a\,\frac{\partial^2}{\partial x^2} - c - p\,\frac{\partial}{\partial t},$$

$a,\,p > 0$, $c \geq 0$, and the right–hand side $f(x,t)$ is sufficiently smooth on the set $\overline{G}$; $\varphi(x,t) = \varphi_{(2.2)}(x,t)$, $(x,t) \in S$.

2.3. Usually, when constructing and studying difference schemes, it is assumed that the problem data are sufficiently smooth, and compatibility conditions are fulfilled on the set S_* that ensure the inclusion $u \in C^{4,2}(\overline{G})$. It is of interest to investigate the convergence of known difference schemes when this inclusion is not fulfilled.

Lowering the smoothness of the solution of the boundary value problem yields the decrease of the convergence rate of the difference schemes that are developed for singularly perturbed problems with sufficiently smooth data. To recover the convergence rate of the difference scheme, it seems appropriate to use the method of special grids that condense in neighbourhoods of both of the boundary and interior layers.

Our aim is to construct difference schemes for the problems (2.2), (2.1) and (2.3), (2.1) that converge ε–uniformly, and to investigate their convergence rate in the cases of nonsmoothness of various types in the initial–boundary conditions such as the discontinuity in the first-order *derivatives* of the initial function with respect to x and of the boundary function with respect to t, and/or the absence of compatibility conditions at the corner points on the set S_*.

The investigation of the problem (2.3), (2.1) which is a model for the problem (2.2), (2.1) allows us to avoid cumbersome technical constructions required in the case of the problem (2.2), (2.1).

3. A Priori Estimates for the Solutions of Problem (2.3)

Here, we obtain some estimates on the solution of the boundary value problem (2.3) and its derivatives. To derive these estimates, we apply the technique developed in [13, 14, 15, 19], where the decomposition of the solution into its regular ("sufficiently" smooth) and singular parts is used. We assume that the functions $f(x,t)$ and $\varphi(x,t)$ are sufficiently smooth on the sets $\overline{G}$ and $\overline{S}^L$, S_0^+, S_0^-, respectively; $\varphi(x,t) \in C(S)$.

3.1. We represent the set $\overline{G}$ as the sum of overlapping subsets

$$\overline{G} = \bigcup_j \overline{G}^j, \quad j = 1, 2, 3, \tag{3.1}$$

where [3]

$$\begin{aligned} &G^1 = G^1(m^1) = \{(x,t):\ |x| < m^1, && t \in (0,T]\},\\ &G^2 = G^2(m^2) = \{(x,t):\ r(x,\, \Gamma) < m^2, && (x,t) \in G\},\\ &G^3 = G^3(m^3) = G \setminus \{G^1(m^3) \textstyle\bigcup G^2(m^3)\}, && m^3 < m^1,\, m^2, \end{aligned}$$

$r(x,\, \Gamma)$ is the distance from the point x to the set Γ, G^1 and G^2 are neighbourhoods of the interior and boundary layers, respectively; let $\overline{G}^1 \bigcap \overline{G}^2 \emptyset$. Sometimes for convenience, we denote by $u^j(x,t)$, $j = 1, 2, 3$, the solution of problem (2.3), (2.1) considered on the set $\overline{G}^j$. Using the results obtained in [13, 14, 15], we establish the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u(x,t) \right| \le M, \quad (x,t) \in \overline{G}^3, \quad k + 2k_0 \le K. \tag{3.2}$$

Here, K is determined by the data of the problem, and $K \ge 4$.

[3] Here and below, m, m^i, m_i (or M, M^i, M_i) denote sufficiently small (or large) positive constants which do not depend on ε and on the discretization parameters.

3.2. Let us investigate the solution of the problem on the set $\overline{G}^{\,1}$. On the set $\overline{G}^{\,1}$, we introduce the function

$$\widehat{u}(x,\,t) = u\big(x,\,t\big)\,\exp(\alpha\,t), \quad (x,\,t) \in \overline{G}^{\,1}, \tag{3.3}$$

where $\alpha = c\,p^{-1}$. In the case of the function $\widehat{u}(x,\,t)$, problem (2.3) considered on the set $\overline{G}^{\,1}$ is transformed into the following problem for the singularly perturbed heat equation

$$L_{(3.4)}\widehat{u}(x,\,t) = \left\{\varepsilon^2\,a\,\frac{\partial^2}{\partial x^2} - p\,\frac{\partial}{\partial t}\right\}\,\widehat{u}(x,\,t) = \widehat{f}(x,\,t),\ (x,\,t) \in G^{\,1}, \tag{3.4a}$$

$$\widehat{u}(x,\,t) = \begin{cases} \widehat{u}^{\,3}(x,\,t),\ (x,\,t) \in S^{\,1} \setminus S, \\ \widehat{\varphi}(x,\,t),\ \ (x,\,t) \in S^{\,1} \bigcap S. \end{cases}$$

Here, $S^{\,1}$ is the boundary of the set $G^{\,1}$, $\overline{G}^{\,1} = G^{\,1} \bigcup S^{\,1}$,

$$\widehat{v}(x,\,t) = v\big(x,\,t\big)\,\exp(\alpha\,t), \tag{3.4b}$$

where $v(x,t)$ is one of the functions $u(x,t)$, $f(x,t)$, $(x,t) \in \overline{G}^{\,1}$, $\varphi(x,t)$, $(x,t) \in S^{\,1} \bigcap \{t=0\}$, $u^{\,3}(x,t)$, $(x,t) \in \overline{G}^{\,1} \bigcap \overline{G}^{\,3}$; $u^{\,3}(x,t) = u(x,t)$, $(x,t) \in \overline{G}^{\,3}$.

We represent the solution of boundary value problem (3.4) as the sum of the functions

$$\widehat{u}(x,\,t) = \widehat{U}(x,\,t) + \widehat{W}(x,\,t), \quad (x,\,t) \in \overline{G}^{\,1}, \tag{3.5a}$$

corresponding to the decomposition

$$u(x,\,t) = U(x,\,t) + W(x,\,t), \quad (x,\,t) \in \overline{G}^{\,1},$$

where $U(x,t)$ and $W(x,t)$ are the regular ("sufficiently" smooth) and singular components of the solution, respectively; $W(x,t)$ is the interior layer function. The function $\widehat{U}(x,t)$ has ε–uniformly bounded derivatives with respect to x up to the Kth order and ε–uniformly bounded derivatives with respect to t up to the $K/2$th order. The function $\widehat{W}(x,t)$ is the solution of the Cauchy problem

$$L_{(3.4)}\widehat{W}(x,\,t) = 0,\ \ (x,t) \in G^{\infty}, \quad \widehat{W}(x,\,t) = \widehat{\Phi}_W(x),\ \ x \in I\!R,\ \ t = 0. \tag{3.6}$$

Here, $G^{\infty} = I\!R \times (0,T]$,

$$\widehat{\Phi}_W(x) = 2^{-1}\sum_{k=1}^{K-1}(k!)^{-1}\left[\frac{\partial^k}{\partial x^k}\,\widehat{\varphi}(0,\,0)\right]|x|\,x^{k-1}, \quad x \in I\!R;$$

$$\left[\frac{\partial^k}{\partial x^k}\,\widehat{\varphi}(0,\,0)\right]\frac{\partial^k}{\partial x^k}\,\widehat{\varphi}(+0,\,0) - \frac{\partial^k}{\partial x^k}\,\widehat{\varphi}(-0,\,0)$$

is the jump of the derivative $\dfrac{\partial^{\,k}}{\partial x^k}\,\widehat{\varphi}(x,\,t)$ at the point $(0,\,0)$.

The function $\widehat{U}(x,\, t)$ is a solution of the problem

$$L_{(3.4)}\widehat{U}(x,\, t) = \widehat{f}(x,\, t), \qquad (x,\, t) \in G^{\,1},$$

$$\widehat{U}(x,\, t) = \begin{cases} \widehat{u}^{\,3}(x,\, t) - \widehat{W}(x,\, t), & (x,\, t) \in S^{\,1},\ t > 0, \\ \widehat{\varphi}(x,\, t) - \widehat{\Phi}_W(x), & (x,\, t) \in S^{\,1},\ t = 0. \end{cases}$$

For the function $\widehat{U}(x,\, t)$, the following estimate holds:

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, \widehat{U}(x,\, t) \right| \le M, \quad (x,\, t) \in \overline{G}^{\,1}, \quad k + 2k_0 \le K, \tag{3.7}$$

which is established taking into account the smoothness of the data of the problem; see [6].

The function $\widehat{W}(x,\, t)$ can be represented as

$$\widehat{W}(x,\, t) = \sum_{k=1}^{K-1} \widehat{W}_k(x,\, t), \quad (x,\, t) \in \overline{G}^{\,\infty}; \tag{3.5b}$$

here, $\widehat{W}_k(x,\, t)$ is a solution of problem (3.6) with $\widehat{\Phi}_W(x)$ defined by

$$\widehat{\Phi}_k(x) = 2^{-1}\,(k!)^{-1} \left[\frac{\partial^k}{\partial x^k}\, \widehat{\varphi}(0,\, 0) \right] |x|\, x^{k-1}, \quad x \in I\!R, \quad k = 1, 2, ..., K-1.$$

The functions $\widehat{W}_k(x,\, t)$ can be written in explicit form; for example, for the function $\widehat{W}_1(x,\, t)$, we have the representation

$$\widehat{W}_1(x,\, t) = 2^{-1} \left[\frac{\partial}{\partial x}\, \widehat{\varphi}(0, 0) \right] \widehat{w}_1(x,\, t), \tag{3.5c}$$

$$\begin{aligned} \widehat{w}_1(x,\, t) &= x\, v(2^{-1}\, \varepsilon^{-1} a^{-1/2}\, p^{1/2}\, x\, t^{-1/2}) \\ &\quad + 2\, \pi^{-1/2}\, \varepsilon\, a^{1/2}\, p^{-1/2}\, t^{1/2}\, \exp(-4^{-1}\, \varepsilon^{-2}\, a^{-1}\, p\, x^2\, t^{-1}), \quad (x, t) \in \overline{G}^{\,\infty}, \end{aligned}$$

$v(x)$ is the error function

$$v(x) = erf(x) = 2\, \pi^{-1/2} \int_0^x \exp(-\alpha^2)\, d\alpha, \quad x \in I\!R.$$

Note that the first derivative of the function $\widehat{W}_1(x,\, t)$ with respect to x is bounded on $\overline{G}^{\,\infty}$ and has a discontinuity at the point $(0,\, 0)$; the first derivatives of the functions $\widehat{W}_k(x,\, t)$, $k \ge 2$, are continuous on $\overline{G}^{\,\infty}$.

It is convenient to represent the function $u(x, t)$, $(x, t) \in \overline{G}^{\,1}$ as the sum of the functions

$$u(x, t) = U^{\,1}(x, t) + W^{\,1}(x, t), \quad (x, t) \in \overline{G}^{\,1}, \tag{3.8a}$$

where $U^{\,1}(x, t)$ and $W^{\,1}(x, t)$ are the regular and singular parts of the solution,

$$U^{1}(x,t)=U^{1}(x,t;\,i)\equiv U(x,t)+\sum_{k=i+1}^{K-1}W_k(x,t),\quad (x,t)\in\overline{G}^{\,1};$$
$$W^{1}(x,t)=W^{1}(x,t;\,i)\equiv\sum_{k=1}^{i}W_k(x,t),\quad (x,t)\in\overline{G}, \tag{3.8b}$$

where $i=1$, if

$$\left[\frac{\partial}{\partial x}\,\varphi(0,0)\right]\neq 0, \tag{3.9}$$

and $i=2$, otherwise. The functions $U(x,t)$ and $W_k(x,t)$ correspond to the components in representations (3.5a), (3.5b); the functions $W_1(x,t)$, $W_2(x,t)$ are defined in $(x,t)\in\overline{G}^{\infty}$ by the relations

$$\begin{aligned}W_1(x,t)&=\widehat{W_1}\,(x,\,t)\,\exp(-\alpha\,t)\\&=2^{-1}\left[\frac{\partial}{\partial x}\,\varphi(0,0)\right]\Big\{x\,v\big(2^{-1}\,\varepsilon^{-1}\,a^{-1/2}\,p^{1/2}\,x\,t^{-1/2}\big)\\&\quad+2\,\pi^{-1/2}\,\varepsilon\,a^{1/2}\,p^{-1/2}\,t^{1/2}\,\exp\big(-4^{-1}\,\varepsilon^{-2}\,a^{-1}\,p\,x^2\,t^{-1}\big)\Big\}\,\exp(-\alpha\,t),\end{aligned} \tag{3.8c}$$

$$\begin{aligned}W_2(x,t)&=\widehat{W_2}\,(x,\,t)\,\exp(-\alpha\,t)\\&=4^{-1}\left[\frac{\partial^2}{\partial x^2}\,\varphi(0,0)\right]\Big\{\big[x^2+2\,\varepsilon^2\,a\,p^{-1}\,t\big]\,v(2^{-1}\,\varepsilon^{-1}\,a^{-1/2}\,p^{1/2}\,x\,t^{-1/2})\\&\quad+2\,\pi^{-1/2}\,\varepsilon\,a^{1/2}\,p^{-1/2}\,x\,t^{1/2}\,\exp(-4^{-1}\,\varepsilon^{-2}\,a^{-1}\,p\,x^2\,t^{-1})\Big\}\,\exp\big(-\alpha\,t\big),\end{aligned}$$

where $\widehat{W_1}(x,t)\widehat{W}_{1(3.5\text{b})}(x,t)$, $\widehat{W_2}(x,t)\widehat{W}_{2(3.5\text{b})}(x,t)$, $\alpha=\alpha_{(3.3)}$.

Taking into account estimates (3.7) and the explicit form of the functions $W_k(x,t)$, we find the following estimates on the components in representation (3.8):

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}\,U^1(x,t)\right|\leq M\left[1+\varepsilon^{\,i+1-k}\,\rho^{\,i+1-k-2k_0}\right],\quad (x,t)\in\overline{G}^{\,1}, \tag{3.10}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}\,W^1(x,t)\right|\leq M\left[1+\varepsilon^{\,i-k}\,\rho^{\,i-k-2k_0}\,\exp(-m\,\varepsilon^{-1}\,|x|)\right],\quad (x,t)\in\overline{G},$$

where $k+2k_0\leq K$, $\rho=\rho(x,t;\,\varepsilon)=\varepsilon^{-1}\,|x|+t^{1/2}$, $i=i_{(3.8\text{b})}$, m is an arbitrary constant.

3.3. Consider the solution of problem (2.3), (2.1) on the set $\overline{G}^{\,2}$. The solution can be represented as the sum of the functions

$$u(x,t)=U(x,t)+V(x,t),\quad (x,t)\in\overline{G}^{\,2}, \tag{3.11}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular components of the solution. The function $U(x,t)$ is the restriction of the function $U^*(x,t)$, $(x,t)\in\overline{G}^{\,2*}$, to the set $\overline{G}^{\,2}$. Here, $U^*(x,t)$ is a solution of the problem

$$\begin{cases} L_{(2.3)}\, U^*(x,t) = f^*(x,t), & (x,t) \in G^{2*}, \\ U^*(x,t) = \varphi^*(x,t), & (x,t) \in S^{2*}, \end{cases} \tag{3.12}$$

where $\overline{G}^{2*} = G^{2*} \cup S^{2*}$. The domain G^{2*} is an extension of the domain G^2 beyond the boundary S^L. The right–hand side of equation (3.12) is a smooth continuation of the function $f(x,t)$. The function $\varphi^*(x,t)$ is smooth on each piecewise smooth parts of the set S^{2*}, and it coincides with the functions $\varphi(x,t)$ and $u^3(x,t)$ on the sets $S^2 \cap S_0$ and $S^2 \cap G^3$, respectively; $\overline{G}^2 = G^2 \cup S^2$. The function $V(x,t)$ is a solution of the problem

$$L_{(2.3)}\, V(x,t) = 0, \qquad (x,t) \in G^2,$$
$$V(x,t) = \begin{cases} \varphi(x,t) - U(x,t), & (x,t) \in S^L \\ 0, & (x,t) \in S^2 \setminus S^L. \end{cases}$$

For simplicity, we assume that compatibility conditions are fulfilled on the set $S_* = S_0 \cap \overline{S}^L$ that ensures the local smoothness of the solution for fixed values of ε [6]; we suppose also that the following inclusion holds on the set $\overline{G}^\delta$, i.e., the δ–neighbourhood of the set S_*:

$$u \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}^\delta), \quad l \geq 2, \quad \alpha \in (0,\,1), \tag{3.13}$$

where δ is a sufficiently small constant. In that case, for the functions $U(x,t)$, $V(x,t)$, we have the estimates for $(x,t) \in \overline{G}^2$, $k + 2k_0 \leq K$:

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U(x,t) \right| \leq M, \tag{3.14a}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, V(x,t) \right| \leq M\, \varepsilon^{-k} \exp\big(- m\, \varepsilon^{-1}\, r(x,\, \Gamma)\big). \tag{3.14b}$$

Here, m is an arbitrary number in the interval $(0,\, m_0)$, where

$$m_0 = a^{-1/2}\, c^{1/2}. \tag{3.15}$$

For example, in order to ensure the inclusion (3.13) for $l = 4$, it suffices to require that the following compatibility conditions are satisfied (compatibility conditions on the derivatives up to the second order with respect to t [6]):

$$\varphi(x^\pm, t) = \varphi(x, t+0), \tag{3.16}$$
$$\left\{ \varepsilon^2\, a\, \frac{\partial^2}{\partial x^2} - c \right\} \varphi(x^\pm, t) - p\, \frac{\partial}{\partial t}\, \varphi(x, t+0) = f(x,t),$$
$$\left\{ \varepsilon^4\, a^2\, \frac{\partial^4}{\partial x^4} - 2\,\varepsilon^2\, a\, c\, \frac{\partial^2}{\partial x^2} + c^2 \right\} \varphi(x^\pm, t) - p^2\, \frac{\partial^2}{\partial t^2}\, \varphi(x, t+0)$$
$$= \left\{ \varepsilon^2\, a\, \frac{\partial^2}{\partial x^2} - c + p\, \frac{\partial}{\partial t} \right\} f(x,t), \quad (x,t) \in S_*,$$

where $x^\pm = x^\pm(x)$, $x^\pm(x) = -d + 0$ for $x = -d$, $x^\pm(x) = d - 0$ for $x = d$.

Theorem 1. *In the boundary value problem* (2.3), (2.1), *assume that* $f \in C^{l_1, l_1/2}(\overline{G})$, $\varphi \in C(S) \cap \{C^{l_1}(S_0^-) \cup C^{l_1}(S_0^+) \cup C^{l_1/2}(\overline{S}^{\,L})\}$, $l_1 = l + \alpha$, $l = K$, $\alpha \in (0,1)$, *and that the condition* (3.13) *is satisfied for the solution of this problem. Then the solution of the boundary value problem and its components in representations* (3.8), (3.11) *satisfy the estimates* (3.2), (3.10), $\{(3.14), 3.15)\}$.

Remark 1. For small values of the parameter ε, x-derivatives of the solution to problem (2.3), (2.1) vary sharply in a neighbourhood of the characteristic S^γ, moreover, in the nearest neighbourhood of S^γ (for $|x| = o(1)$), the first derivative with respect to x is bounded whereas the higher derivatives with respect to x grow unboundedly for $|x| = \mathcal{O}(\varepsilon\, t^{1/2})$, $\varepsilon\, t^{1/2} \to 0$, and they become bounded for $\varepsilon^{-1}\, t^{-1/2}\, |x| \to \infty$, $\varepsilon\, t^{1/2} = \mathcal{O}(1)$. The interior layer, i.e., the *transient parabolic layer*, is a solution of the boundary value problem in a neighbourhood of S^γ, out of which the derivatives with respect to x are bounded. The interior layer generated by the piecewise smooth initial function is weak.

From estimates (3.11), the next estimate follows

$$\left| \frac{\partial}{\partial t}\, u(x,t) \right| \le M \left[\varepsilon^{-1}\, |x| + t^{1/2} \right]^{-1} \quad \textit{for} \;\; \varepsilon^{-1}\, |x| + t^{1/2} \le m,$$

i.e., the derivative $\dfrac{\partial}{\partial t}\, u(x,t)$ is not bounded in a neighbourhood of the point $(0,0)$. Thus, the singularity of the solution that is generated by the discontinuity in the first derivative of the function $\varphi\,(x,t)$ with respect to x at the point $(0,0)$ turns out to be essentially stronger than its singularity generated by the discontinuity in the first derivative of the function $\varphi\,(x,t)$ with respect to t on $S^{\,L}$. □

3.4. In the case of problem (2.2), (2.1), the function $W^{\,1}(x,t)$ in the representation of its solution on $\overline{G}^{\,1}$ in the form (3.8a) is defined by the relation

$$W^{\,1}(x,t) = \sum_{k=1}^{i} W_k(x,t), \quad (x,t) \in \overline{G}^{\,1}, \quad i = i_{(3.8b)}. \tag{3.17a}$$

The functions $W_1(x,\, t)$, $W_2(x,\, t)$ that are defined by relations (3.8c) are solutions of the problems

$$L_{(3.17b)}\, W_1(x,\, t) \equiv \left\{ \varepsilon^2\, a_0(t)\, \frac{\partial^2}{\partial x^2} - c_0(t) - p_0(t)\, \frac{\partial}{\partial t} \right\} W_1(x,\, t) = 0, \quad (x,\, t) \in G^{\infty},$$

$$W_1(x,\, 0) = 2^{-1} \left[\frac{\partial}{\partial x}\, \varphi(0,0) \right] |x|, \quad x \in I\!\!R; \tag{3.17b}$$

$$L_{(3.17b)}\, W_2(x,\, t) = F_1(x,\, t), \quad (x,\, t) \in G^{\infty},$$

$$W_2(x,\, 0) = 4^{-1} \left[\frac{\partial^2}{\partial x^2}\, \varphi(0,0) \right] x\, |x|, \quad x \in I\!\!R.$$

Here $G^{\infty} = G^{\infty}_{(3.6)}$,

$$F_1(x,\, t) = -x \left\{ \varepsilon^2\, a_1(t)\, \frac{\partial^2}{\partial x^2} - c_1(t) - p_1(t)\, \frac{\partial}{\partial t} \right\} W_1(x,\, t), \quad (x,\, t) \in G^{\infty},$$

$$v_0(t) = v(0,\, t), \quad v_1(t) = \frac{\partial}{\partial x}\, v(0,\, t),$$

where $v(x,\, t)$ is one of the functions $a(x,\, t),\, \ldots,\, p(x,\, t)$.

In the case of problem (2.2), (2.1), we have the following result, which is similar to Theorem 1 (see [19]).

Theorem 2. *In the boundary value problem* (2.2), (2.1), *assume that* a, c, p, $f \in C^{\,l_1, l_1/2}(\overline{G})$, $\varphi \in C(S) \cap \{C^{\,l_1}(S_0^-) \cup C^{\,l_1}(S_0^+) \cup C^{\,l_1/2}(\overline{S}^{\,L})\}$, $l_1 = l + \alpha$, $l = K$, $\alpha \in (0,1)$, *and that the condition* (3.13) *is satisfied for the solution of this problem. Then the solution of the boundary value problem and its components in representations* (3.8a), (3.17), (3.11), *satisfy the estimates* (3.2), (3.10), (3.14)*; the constant* m *in* (3.14b) *is an arbitrary one in the interval* $(0,\, m_0)$, *where*

$$m_0 = \min_{\overline{G}} \left[a^{-1/2}(x,t)\, c^{1/2}(x,t) \right]. \tag{3.18}$$

Remark 2. In the case when condition (3.13) in Theorem 2 is violated, the component $V(x,\, t)$, $(x,\, t) \in \overline{G}^{\,2}$, satisfies the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, V(x,t) \right| \le M\, \varepsilon^{-k} \left[1 + \rho^{\,2-k-2\,k_0} \right] \exp\left(-\, m\, \varepsilon^{-1}\, r(x,\, \Gamma) \right), \tag{3.19}$$

where $m \in (0,\, m_0)$, $m_0 = m_{0(3.18)}$, $\rho = \rho(x, t;\, \varepsilon) = \varepsilon^{-1}\, r(x,\, \Gamma) + t^{1/2}$.

This statement is valid also for Theorem 1 provided that $m_0 = m_{0(3.15)}$.

4. Classical Approximations of Problem (2.3), (2.1) on Uniform Grids

Let us construct a difference scheme based on the classical approximation of the boundary value problem (2.3), (2.1) when initial and boundary functions are piecewise smooth, and they are only continuous on S_* (in this case, estimates (3.2), (3.10), (3.14a), $\{(3.19), (3.15)\}$ take place). We study the convergence of this scheme for not too small values of the parameter ε compared to the step–size of the uniform mesh with respect to x. The technique used to construct and investigate these schemes is similar to that used in [19] for the parabolic convection–diffusion equation (see also [2, 8, 13, 14, 15, 19]).

4.1. On the set $\overline{G}_{(2.1)}$, we introduce the rectangular grid

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega} \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are meshes on the intervals $[-d,\, d]$ and $[0,\, T]$, respectively. The mesh $\overline{\omega}$ has an arbitrary distribution of nodes satisfying only the condition

$h \le MN^{-1}$, where $h = \max_i h^i$, $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}$. The mesh $\overline{\omega}_0$ is uniform with the step–size $h_0 = TN_0^{-1}$. Here, $N+1$ and N_0+1 are the numbers of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively.

We approximate boundary value problem (2.3) by the finite difference scheme [10]

$$\begin{cases} \Lambda_{(4.2)}\, z(x,t) = f(x,t), & (x,t) \in G_h, \\ z(x,t) = \varphi(x,t), & (x,t) \in S_h. \end{cases} \tag{4.2}$$

Here $\Lambda_{(4.2)} \equiv \varepsilon^2 a \delta_{\overline{x}\widehat{x}} - c - p\delta_{\overline{t}}$,

$$\delta_{\overline{x}\widehat{x}} z(x,t) z_{\overline{x}\widehat{x}}(x,t) = 2(h^i + h^{i-1})^{-1}[\delta_x z(x,t) - \delta_{\overline{x}} z(x,t)],$$

$(x,t) = (x^i,t) \in G_h$, is the second difference derivative on a nonuniform mesh, $\delta_x z(x,t)$ and $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}}\, z(x,t)$ are the first (forward and backward) difference derivatives,

$$\delta_x z(x,t) = \left(h^i\right)^{-1}\left(z(x^{i+1},t) - z(x^i,t)\right),$$
$$\delta_{\overline{x}} z(x,t) = \left(h^{i-1}\right)^{-1}\left(z(x^i,t) - z(x^{i-1},t)\right),$$
$$\delta_{\overline{t}}\, z(x,t) = \tau^{-1}\left(z(x^i,t) - z(x^i,t-\tau)\right).$$

Difference scheme (4.2), (4.1) is ε–uniformly monotone (see [10]). The following version of the comparison theorem holds.

Theorem 3. *Let the functions $z^1(x,t)$, $z^2(x,t)$, $(x,t) \in \overline{G}_h$ satisfy the conditions*

$$\Lambda\, z^1(x,t) < \Lambda\, z^2(x,t), \quad (x,t) \in G_h, \quad z^1(x,t) > z^2(x,t), \quad (x,t) \in S_h.$$

Then $z^1(x,t) > z^2(x,t)$, $(x,t) \in \overline{G}_h$.

Consider scheme (4.2) on the uniform grid

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0. \tag{4.3}$$

Let, for the initial function $\varphi(x,t)$, condition (3.9) be satisfied. Using *a priori* estimates (3.2), (3.10) for $i=1$, (3.14) for the solutions of problem (2.3) and the majorant function technique from [10], and using reasoning similar to that employed in [13, 14, 15, 19], we find the estimate

$$|u(x,t) - z(x,t)| \le M\left[N^{-1} + (\varepsilon + N^{-1})^{-2} N^{-2} \ln(M_1 + \varepsilon N) + \varepsilon N_0^{-1/2} + N_0^{-1} \ln N_0\right], \quad (x,t) \in \overline{G}_h. \tag{4.4}$$

Here we used the majorant function $w(x,t)$ satisfying the condition

$$\Lambda w(x,t) \le -M\Bigg\{\varepsilon \min\Big[\max_{\substack{x\in\overline{D} \\ (x,t)\notin S_* \\ (x,t)\ne(0,0)}} \Big|\frac{\partial^2}{\partial x^2} u(x,t)\Big|, N^{-2} \max_{\substack{x\in\overline{D} \\ (x,t)\notin S_* \\ (x,t)\ne(0,0)}} \Big|\frac{\partial^4}{\partial x^4} u(x,t)\Big|\Big]$$
$$+ \min\Big[\max_{\substack{x\in\overline{D} \\ (x,t)\notin S_* \\ (x,t)\ne(0,0)}} \Big|\frac{\partial}{\partial t} u(x,t)\Big|, N_0^{-1} \max_{\substack{x\in\overline{D} \\ (x,t)\notin S_* \\ (x,t)\ne(0,0)}} \Big|\frac{\partial^2}{\partial t^2} u(x,t)\Big|\Big]\Bigg\}, \quad (x,t) \in \overline{G}.$$

Thus, scheme (4.2), (4.3) converges under the condition $N^{-1} = o(\varepsilon)$; for fixed values of the parameter ε, the scheme converges at the rate $\mathcal{O}(N^{-1} + N_0^{-1/2})$.

Let *a priori* estimates (3.2), (3.10), (3.14a), {(3.19), (3.15)} be fulfilled for $K = 4$, and let the component $W_1(x,t)$ in representation (3.8) vanishes, i.e.,

$$W_1(x,t) = 0, \quad (x,t) \in \overline{G}^1. \tag{4.5}$$

In that case, the following relation holds:

$$\left[\frac{\partial}{\partial x}\,\varphi(0,0)\right] = 0, \tag{4.6}$$

i.e., the derivative $(\partial/\partial x)\,\varphi(x,t)$ is continuous on S_0. For scheme (4.2), (4.3), taking into account the *a priori* estimates ($i = 2$ in (3.10)), we obtain the estimate for $(x,t) \in \overline{G}_h$:

$$|u(x,t) - z(x,t)| \le M\left[(\varepsilon + N^{-1})^{-2}\,N^{-2}\,\ln(M_1 + \varepsilon\,N) + N_0^{-1}\,\ln N_0\right], \tag{4.7}$$

the finite difference scheme converges for fixed values of the parameter ε at the rate $\mathcal{O}(N^{-2}\ln N + N_0^{-1}\ln N_0)$.

But if the component $V(x,t)$ in representation (3.11) vanishes, i.e.,

$$V(x,t) = 0, \quad (x,t) \in \overline{G}^2, \tag{4.8}$$

(the fulfillment of condition (4.5) is not assumed), we have the estimate

$$|u(x,t) - z(x,t)| \le M\left[N^{-1} + N_0^{-1} + \varepsilon\,N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h. \tag{4.9}$$

Thus, under condition (4.8), scheme (4.2), (4.3) converges ε–uniformly at the rate $\mathcal{O}(N^{-1} + N_0^{-1/2})$.

In that case when both conditions (4.5) and (4.8) are fulfilled (we denote this case by {(4.5), (4.8)}), the following estimate holds for $(x,t) \in \overline{G}_h$:

$$|u(x,t) - z(x,t)| \le M\left[N^{-2}\ln(M_1 + \varepsilon N) + N_0^{-1} + \varepsilon^2 N_0^{-1}\ln N_0\right], \tag{4.10}$$

i.e. scheme (4.2), (4.3) converges ε–uniformly at the rate $\mathcal{O}\big(N^{-2}\ln N + N_0^{-1}\ln N_0\big)$.

The following theorem takes place.

Theorem 4.1. *Let the estimates* (3.2), (3.10), (3.14a), {(3.19), (3.15)} *with* $K = 4$ *be satisfied for the solution of problem* (2.3), (2.1) *and its components in representation* (3.8), (3.11). *Then the difference scheme* (4.2), (4.3) *converges under the condition* $N^{-1} = o(\varepsilon)$. *Under condition* (4.8), *the scheme* (4.2), (4.3) *converges* ε*-uniformly. The discrete solutions satisfy the estimate* (4.4); *and, in the case of conditions* (4.5), (4.8) *and* {(4.5), (4.8)}, *they satisfy the estimates* (4.7), (4.9) *and* (4.10), *respectively.*

Remark 3. From estimates (4.4), (4.7) it follows that the discontinuity of the first-order derivative with respect to x of the initial function leads to a decrease of the convergence rate of the scheme (4.2), (4.3) for fixed values of the parameter ε. Under this, the convergence order decreases two times up to a logarithmic factor.

4.2. Let us give estimates for solutions of the difference scheme (4.2), (4.3) in the case when the first-order derivative with respect to t of the function $\varphi(x,t)$ has a jump discontinuity on the set S^L, and when *weakened compatibility conditions* are given on the set S_* compared with the condition (3.17) required usually to ensure the inclusion $u \in C^{4,2}(\overline{G})$.

Let the boundary function $\varphi(x,t)$ be sufficiently smooth on the sets S_0 and $\overline{S}^L$, and let a compatibility condition be satisfied on the set S_* for the data $\varphi(x,t)$, $(x,t) \in S$ and $f(x,t)$, $(x,t) \in \overline{G}$ that ensures only the continuity of the derivative $(\partial/\partial t)\, u(x,t)$ on S_* (i.e., a compatibility condition for the first-order derivative with respect to t is satisfied; see [6])

$$\varphi(x^{\pm},t) = \varphi(x,t+0), \tag{4.11}$$

$$\left\{\varepsilon^2\, a\, \frac{\partial^2}{\partial x^2} - c\right\} \varphi(x^{\pm},t) - p\, \frac{\partial}{\partial t}\, \varphi(x,t+0) = f(x,t), \quad (x,t) \in S_*,\ x^{\pm} = x^{\pm}_{(3.16)}.$$

We call problem (2.3), (2.1) with such data *the standard problem*. The derivative $(\partial^2/\partial t^2)\, u(x,t)$ on S_* for the standard problem, in general, is discontinuous; $u \notin C^{4,2}(\overline{G})$. In this case, for the solution of difference scheme (4.2), (4.3), we obtain the following estimate that is similar to estimate (4.7):

$$|u(x,t) - z(x,t)| \le M\left[(\varepsilon + N^{-1})^{-2}\, N^{-2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h. \tag{4.12}$$

Thus – even under the weakened condition (4.11) – the estimate of the convergence rate is the same as it is under the condition (3.17).

Let the boundary function be sufficiently smooth on the set S_0, and be piecewise smooth on the set $\overline{S}^L$, and assume that the first-order derivative with respect to t of the function $\varphi(x,\, t)$ has a discontinuity at the point $(x_0,\, t_0) \in S^L$, $t_0 \in (0,\, T]$, i.e.,

$$\left[\frac{\partial}{\partial t}\, \varphi(x_0,\, t_0)\right] \ne 0, \quad (x_0,\, t_0) \in S^L, \tag{4.13}$$

and let only the continuity condition be imposed for the function $\varphi(x,\, t)$ on the set S_*:

$$\varphi(x^{\pm},t) = \varphi(x,t+0), \quad (x,t) \in S_*, \quad x^{\pm} = x^{\pm}_{(3.16)}(x). \tag{4.14}$$

The fulfillment of a compatibility condition for the first-order derivative with respect to t is not assumed; in this case, the derivative $(\partial/\partial t)\, u(x,t)$ on the set S_*, in general, is discontinuous [6]. Then, for the discrete solution, we obtain estimate (4.7), i.e., the same one as (4.12) up to logarithmic factors.

5. A Difference Scheme for Problem (2.3), (2.1) on Piecewise Uniform Meshes

Let us consider the behaviour of solutions to difference scheme (4.2), (4.1) when piecewise uniform meshes are used, and various types of nonsmoothness in the initial–boundary function $\varphi(x,t)$ take place.

5.1. On the set $\overline{G}$, we construct the grid condensing in a neighbourhood of the boundary layer; this grid is similar to that constructed in [2, 8, 13, 14, 15, 19]

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega}^* \times \overline{\omega}_0, \qquad (5.1a)$$

where $\overline{\omega}_0 = \overline{\omega}_{0(4.1)}$, $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a piecewise uniform mesh on $[-d,\, d]$, and σ is a parameter depending on ε and N. We choose σ so as to satisfy the condition

$$\sigma = \sigma(N,\, \varepsilon) = \min\,[\,\beta,\, 2\,m^{-1}\,\varepsilon\, \ln N\,], \qquad (5.1b)$$

where β is an arbitrary number in the interval $(0,\, d)$, $m \in (0,\, m_{0(3.15)})$. The interval $[-d,\, d]$ is divided into three parts: $[-d,\, -d+\sigma]$, $[-d+\sigma,\, d-\sigma]$ and $[d-\sigma,\, d]$; in each part, the mesh step-size is constant and is equal to $h^{(1)} = 2\,d\,\sigma\,\beta^{-1}\,N^{-1}$ on the intervals $[-d, -d+\sigma]$, $[d-\sigma,\, d]$ and to $h^{(2)} = 2\,d\,(\,d-\sigma)\,(\,d-\beta)^{-1}\,N^{-1}$ on the interval $[-d+\sigma,\, d-\sigma]$, $\sigma = \sigma_{(5.1)}$.

We call the difference scheme (4.2) on piecewise uniform mesh (5.1) the basic scheme for problem (2.3), (2.1).

5.2. In that case when the function $\varphi(x,t)$ satisfies the condition (3.9), and taking into account *a priori* estimates of the boundary value problem (2.3), we obtain the following estimate for the solution of the basic scheme:

$$|\, u(x,t) - z(x,t)\,| \le M\, \left[\, N^{-1} + \varepsilon\, N_0^{-1/2} + N_0^{-1}\, \ln N_0 \right], \quad (x,t) \in \overline{G}_h; \quad (5.2a)$$

and the following ε-uniform estimate also holds:

$$|\, u(x,t) - z(x,t)\,| \le M\, \left[\, N^{-1} + N_0^{-1/2} \right], \quad (x,t) \in \overline{G}_h. \qquad (5.2b)$$

In that case when the initial function $\varphi(x,t)$ satisfies condition (4.6), we have the estimate

$$|\, u(x,t) - z(x,t)\,| \le M\, \left[\, N^{-2}\, \ln^3 N + N_0^{-1}\, \ln N_0 \,\right], \quad (x,t) \in \overline{G}_h. \qquad (5.3)$$

Theorem 4. *Let the hypotheses of Theorem* 4.1 *be fulfilled. Then, the solution of the basic difference scheme* (4.2), (5.1) *converges ε-uniformly. The discrete solutions satisfy estimate* (5.2)*; and, under condition* (4.6)*, they satisfy estimate* (5.3).

5.3. We expose estimates of the discrete solutions in the case of a discontinuity in the first derivative of the function $\varphi(x,t)$, $(x,t) \in S^L$, with respect to t, and under weakened compatibility conditions given on the set S_*.

For the standard problem (see Subsection 4.2), we have the estimate

$$|\, u(x,t) - z(x,t)\,| \le M\, \left[\, N^{-2}\, \ln^2 N + N_0^{-1} \,\right], \quad (x,t) \in \overline{G}_h. \qquad (5.4)$$

But if the initial-boundary function $\varphi(x,t)$ is smooth on S_0, and is piecewise smooth on $\overline{S}^L$ (for the function $\varphi(x,\, t)$, condition (4.13) holds), moreover, for the function $\varphi(x,\, t)$ on the set S_*, the continuity condition (4.14) be imposed only, then we obtain the estimate (5.3) for the discrete solution, i.e., the estimate (5.4) up to the logarithmic factor with respect to N_0. Thus, the absence of compatibility conditions on S_* (except the continuity of $\varphi(x, t)$ on S_*), and also discontinuities of the derivatives $(\partial/\partial t)\,\varphi(x,t)$ on S^L do not significantly worsen the estimate (5.4); estimates (5.3) and (5.4) differ only by logarithmic factors acting to N^{-2} and N_0^{-1}.

From comparison of the estimates (5.2) and (5.4), it follows that the discontinuity of the first-order derivative with respect to x of the initial function leads to an essential decrease of the convergence rate for the special scheme (4.2), (5.1). When the initial function is piecewise smooth, and it satisfies the condition $\left[\frac{\partial}{\partial x}\,\varphi(0,0)\right] \neq 0$, then, the order of the ε-uniform convergence rate of the scheme decreases two times up to the logarithmic factor acting to N.

5.4. Let condition (3.9) hold for the function $\varphi(x,t)$. Under the hypotheses of Theorem 4.1, the use of meshes that condense in both boundary and interior layers allows us to improve the scheme convergence for small values of the parameter ε.

Let us consider scheme (4.2) on the mesh

$$\overline{G}_h = \overline{\omega}^{\,**} \times \overline{\omega}_0, \tag{5.5a}$$

that condense in neighbourhoods of the sets S^L and S^γ, i.e., in neighbourhoods of the boundary and interior layers. In $\overline{G}_{h(5.5a)}$, $\overline{\omega}_0 = \overline{\omega}_{0(4.1)}$, $\overline{\omega}^{\,**} = \overline{\omega}^{\,**}(\sigma)$ is a piecewise uniform mesh; we choose the value σ satisfying the condition

$$\sigma = \min\left[\beta,\, 2\,m^{-1}\,\varepsilon\,\ln N\right], \tag{5.5b}$$

where $m = m_{(5.1)}$, β is an arbitrary number in the interval $(0,\, 2^{-1}\,d)$. The mesh $\overline{\omega}^{\,**}$ is constructed to be symmetric with respect to the middle of the interval $[-d,\, d]$. The interval $[0\, d]$ is divided into three parts: $[0,\, \sigma]$, $[\sigma,\, d-\sigma]$ and $[d-\sigma,\, d]$. In each part, the mesh step-size is constant and equal to $h^{(1)} = 2\,d\,\sigma\,\beta^{-1}\,N^{-1}$ on the intervals $[0,\, \sigma]$, $[d-\sigma,\, d]$ and to $h^{(2)} = 2\,d\,(d-2\,\sigma)\,(d-2\,\beta)^{-1}\,N^{-1}$ on the interval $[\sigma,\, d-\sigma]$.

We call scheme (4.2), (5.5) *the improved difference scheme* for problem (2.3), (2.1). When the hypotheses of Theorem 4.1 are fulfilled, for the improved difference scheme, we obtain the estimate

$$\begin{aligned} |\, u(x,t) - z(x,t)\,| \leq M\, \big[\, N^{-2}\,\ln^2 N + \varepsilon\, N^{-1}\, \min[\varepsilon\,\ln N,\, 1] \\ + N_0^{-1}\,\ln N_0 + \varepsilon\, N_0^{-1/2}\,\big], \quad (x,t) \in \overline{G}_h; \end{aligned} \tag{5.6}$$

this estimate is obtained similarly to (4.4). For small values of the parameter ε, the convergence rate of the improved difference scheme (4.2), (5.5) is better than that for the scheme (4.2), (4.3), however, for fixed values of the parameter ε, the convergence rate of these schemes is the same.

6. A Difference Scheme for Problem (2.2), (2.1)

We consider the problem (2.2), (2.1), assuming that initial-boundary conditions of this problem are the same as those for the problem (2.3), (2.1). The construction and investigation of a difference scheme for problem (2.2), (2.1) is carried out similarly to those in Sections 4, 5.

6.1. The problem (2.2), (2.1) is approximated by the finite difference scheme [10]

$$\begin{cases} \Lambda_{(6.1)}\, z(x,t) = f(x,t), & (x,t) \in G_h, \\ z(x,t) = \varphi(x,t), & (x,t) \in S_h. \end{cases} \tag{6.1}$$

Here $\overline{G}_h = \overline{G}_{h(4.1)}$,

$$\Lambda_{(6.1)} \equiv \varepsilon^2\, a(x,t)\, \delta_{\overline{x}\widehat{x}} - c(x,t) - p(x,t)\, \delta_{\overline{t}}.$$

For solutions of the difference scheme (6.1) on uniform mesh (4.3), the same estimates are valid as for the scheme (4.2), (4.3) (see Subsection 4.1).

When constructing an ε-uniform convergent scheme, we use the piecewise uniform mesh

$$\overline{G}_h = \overline{G}_{\mathrm{h}(5.1\mathrm{a})}(\sigma), \quad \sigma = \sigma_{(5.1\mathrm{b})}(m), \tag{6.2}$$

where m is an arbitrary constant in the interval $(0,\, m_0)$, $m_0 = m_{0(3.18)}$. We call the difference scheme (6.1) on piecewise uniform mesh (6.2) the basic scheme for problem (2.2), (2.1).

For the basic scheme (6.1), (6.2), ε-uniform estimates are valid that are similar to those for the basic scheme (4.2), (5.1) for problem (2.3), (2.1).

Theorem 5. *Let the solution of problem* (2.2), (2.1) *and its components in representations* (3.8a), (3.17), (3.11) *satisfy estimates* (3.2), (3.10), (3.14a), $\{(3.19), (3.18)\}$ *for* $K = 4$. *Then, the basic difference scheme* (6.1), (6.2) *converges* ε*-uniformly, whereas the scheme* (6.1) *on the uniform mesh* (4.3) *converges under the condition* $N^{-1} = o(\varepsilon)$. *For solutions of the basic difference scheme* (6.1), (6.2) $\big($*scheme* (6.1), (4.3)$\big)$, *estimates* (5.2) $\big($*the estimate* (4.4)$\big)$ *are valid, and, in the case of condition* (4.6) $\big($*conditions* (4.6), (4.8) *and* $\{(4.6), (4.8)\}\big)$, *the estimate* (5.3) *holds* $\big($*estimates* (4.7), (4.9) *and* (4.10) *hold, respectively*$\big)$.

6.2. In the case of difference schemes (6.1), (4.3) and (6.1), (6.2), results are valid that are similar to those obtained in Subsections 4.2, 5.3 and 5.4. In particular, under condition (3.9), for solutions of the basic difference scheme (6.1), (6.2) estimate (5.2b) holds; in the case of condition (4.6), we have estimate (5.3). In estimates (5.2b), (5.3), the function $u(x,t)$ is the solution of problem (2.2), (2.1).

Let the boundary function $\varphi(x,t)$ be sufficiently smooth on the sets S_0 and $\overline{S}^L$, and let the compatibility conditions be given on the set S_* :

$$\varphi(x^{\pm},t) = \varphi(x,t+0), \tag{6.3}$$

$$\Big\{\varepsilon^2\, a(x,t)\, \frac{\partial^2}{\partial x^2} - c(x,t)\Big\}\, \varphi(x^{\pm},t) - p(x,t)\, \frac{\partial}{\partial t}\, \varphi(x,t+0) = f(x,t), \quad (x,t) \in S_*,$$

where $x^{\pm} = x^{\pm}_{(3.16)}(x)$. We call problem (2.2), (2.1) with such data the *standard problem*. The solution of the basic difference scheme (6.1), (6.2), in the case of the standard problem, satisfies the estimate (5.4).

Let the condition (3.9) be satisfied for the function $\varphi(x,t)$. Consider scheme (6.1) on the mesh condensing in neighbourhoods of boundary and interior layers:

$$\overline{G}_h = \overline{G}_{\mathrm{h}(5.5\mathrm{a})}(\sigma), \quad \sigma = \sigma_{(5.5\mathrm{b})}(m), \tag{6.4}$$

where m is an arbitrary constant in the interval $(0,\, m_0)$, $m_0 = m_{0(3.18)}$. Difference scheme (6.1), (6.4) is the *improved difference scheme* for the problem (2.2), (2.1).

For the solution of the improved difference scheme, estimate (5.6) holds. From this estimate, it follows that, under the condition

$$\varepsilon \leq M\,[N^{-1} + N_0^{-1/2}], \tag{6.5a}$$

the following estimate takes place:

$$|\, u(x,t) - z(x,t) \,| \leq M\,\left[N^{-2}\ln^2 N + N_0^{-1}\ln N_0\right], \quad (x,t) \in \overline{G}_h. \tag{6.5b}$$

Where the ε–uniform convergence is attained only when the value of the parameter ε is bounded by a function of N and/or N_0, we say that the scheme converges *conditionally ε–uniformly*; otherwise, we say that the scheme converges *unconditionally ε–uniformly* (or, shortly, ε–uniformly).

Thus, the use of meshes (6.4) instead of (6.2), in general, does not improve the unconditional ε-uniform convergence rate of scheme (6.1). However, under condition (6.5a), scheme (6.1), (6.4) converges *conditionally ε-uniformly* at the rate $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-1}\ln N_0)$, i.e., with the rate that is considerably better than that for basic scheme (6.1), (6.2).

Remark 4. Grid approximations to singularly perturbed parabolic convection-diffusion equations in the presence of a discontinuity of the first derivative of the initial function with respect to x have been considered in [7, 19]; approximations to solutions and their first derivatives (in [7]), and the derivatives $(\partial^{k+k_0}/\partial x^k \partial t^{k_0})\, u(x,t)$, $k + 2k_0 \leq 2$ (in [19]) have been studied in a neighbourhood of the moving interior layer. Theoretical investigations [7, 19], and numerical experiments [7] have been shown that discrete solutions on the mesh that is uniform in a neighbourhood of the interior layer, converge ε-uniformly at the rate $\mathcal{O}(N^{-1/2} + N_0^{-1/2})$. The observable (in [7, 19]) decrease of the convergence rate with respect to x compared to the estimate (5.2b) is caused by the presence of the convective term in the differential equation.

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