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On Strong Convergence of Halpern's Method for Quasi-Nonexpansive Mappings in Hilbert Spaces

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Abstract. In this paper, we introduce a Halpern's type method to approximate common fixed points of a nonexpansive mapping T and a strongly quasi-nonexpansive mappings S, defined in a Hilbert space, such that $I - S$ is demiclosed at 0. The result shows as the same algorithm converges to different points, depending on the assumptions of the coefficients. Moreover, a numerical example of our iterative scheme is given.

Keywords: approximation algorithm, fixed point, variational inequality.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, which induces the norm $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H. Let T be a mapping of C into itself; we denote by $Fix(T)$ the set of fixed points of T, that is, $Fix(T) = \{z \in C : Tz = z\}$.

We recall that a mapping $T : C \to H$ is said to be nonexpansive if $\forall x, y \in C$, the following inequality holds

$$\|Tx - Ty\| \leq \|x - y\|.$$

If $T : C \to H$ is a mapping with $Fix(T) \neq \emptyset$, then T is said to be quasi-nonexpansive if, $\forall x \in C, \forall p \in Fix(T)$,

$$\|Tx - p\| \leq \|x - p\|.$$

Further, the set of fixed points of a quasi-nonexpansive mapping is closed and convex [9].

The problem to approximate fixed points of nonexpansive mappings has been widely investigated by many authors. In the setting of Banach spaces, in 2008, F. Kohsaka and W. Takahashi [10] defined the concept of nonspreading mappings. In the setting of Hilbert spaces, the following characterization of a nonspreading mapping was proved by S. Iemoto and W. Takahashi [8] in 2009. $T : C \to C$ as a nonspreading mapping if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{1.1}$$

Observe that if T is a nonspreading mapping from C into itself and $Fix(T) \neq \emptyset$, then T is quasi-nonexpansive.

Recently, T. Suzuki [16] introduced the concept of Chatterjea mapping.

Let T be a mapping on a subset C of a Banach space E and let η be a continuous strictly increasing function from $[0, \infty)$ into itself with $\eta(0) = 0$. Then T is called a Chatterjea mapping with respect to η if

$$2\eta(\|Tx - Ty\|) \leq \eta(\|Tx - y\|) + \eta(\|x - Ty\|), \quad \forall x, y \in C. \tag{1.2}$$

It is easy to check that a nonspreading mapping on a subset C of a Hilbert space E is a Chatterjea mapping with respect to the function $t \mapsto t^2$.

S. Iemoto and W. Takahashi [8] approximated common fixed points of a nonexpansive mapping T and of a nonspreading mapping S in a Hilbert space using Moudafi's iterative scheme [13]. They obtained the following theorem that shows the weak convergence of their iterative method:

Theorem 1. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Assume that $Fix(S) \cap Fix(T) \neq \emptyset$. Define a sequence (x_n) as follows:*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[\beta_n S x_n + (1 - \beta_n)T x_n], \end{cases} \tag{1.3}$$

where (α_n), (β_n) are in [0,1]. Then, the following hold:

(i) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then (x_n) converges weakly to $p \in Fix(S)$;

(ii) If $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then (x_n) converges weakly to $p \in Fix(T)$;

(iii) If $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$, then (x_n) converges weakly to $p \in Fix(S) \cap Fix(T)$.

In order to overcome the weak convergence in [8], in [6] the authors modified the algorithm (1.3) in a Halpern's type method, using the averaged type mappings T_δ, i.e. mapping:

$$T_\delta = (1-\delta)I + \delta T, \quad \delta \in (0,1).$$

The main theorem of [6] is given below.

Theorem 2. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and let $S : C \to C$ be a nonspreading mapping such that $Fix(S) \cap Fix(T) \neq \emptyset$. Let T_δ and S_δ be the averaged type mappings. Suppose that (α_n) is a real sequence in $(0,1)$ satisfying the conditions:*

$$1.\ \lim_{n\to\infty} \alpha_n = 0, \qquad 2.\ \sum_{n=1}^{\infty} \alpha_n = \infty.$$

If (β_n) is a sequence in $[0,1]$, we define a sequence (x_n) as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n) S_\delta x_n], \end{cases} \tag{1.4}$$

then, the following hold:

(i) If $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then (x_n) strongly converges to $\overline{p} = P_{Fix(T)}u$ which is the unique solution in $Fix(T)$ of the variational inequality $\langle u-\overline{p}, x-\overline{p}\rangle \leq 0$, for all $x \in Fix(T)$.

(ii) If $\sum_{n=1}^{\infty} \beta_n < \infty$, then (x_n) strongly converges to $\widehat{p} = P_{Fix(S)}u$ which is the unique solution in $Fix(S)$ of the variational inequality $\langle u-\widehat{p}, x-\widehat{p}\rangle \leq 0$, for all $x \in Fix(S)$.

(iii) If $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then (x_n) strongly converges to $p_0 = P_{Fix(T)\cap Fix(S)}u$ which is the unique solution in $Fix(T)\cap Fix(S)$ of the variational inequality $\langle u-p_0, x-p_0\rangle \leq 0$, for all $x \in Fix(T)\cap Fix(S)$.

The aim of this paper will be to improve the Theorem 2 without using averaged type mappings.

2 Main Result

In our result we need of the concept of strongly quasi-nonexpansive mappings.

We recall that the concept of strongly nonexpansive mapping was introduced by Bruck and Reich in 1977 [5], as follows: a mapping T is said strongly nonexpansive if T is nonexpansive and whenever $(x_n - y_n)$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$, it follows that $(x_n - y_n) - (Tx_n - Ty_n) \to 0$.

To our knowledge, Saejung [15] in 2010 introduced the concept of strong quasi-nonexpansivity: a mapping S is said strongly quasi-nonexpansive if $Fix(S) \neq \emptyset$, S is quasi-nonexpansive and $x_n - Sx_n \to 0$ whenever (x_n) is a bounded sequence such that $\|x_n - p\| - \|Sx_n - p\| \to 0$ for some $p \in Fix(S)$.

In [5] it was proved that an averaged mapping of a nonexpansive mapping defined on a uniformly convex Banach space is strongly nonexpansive.

Remark 1. Following the same line on the proof in [5], one can show that an averaged type mapping $S_\delta = (1-\delta)I + \delta S$ of a quasi-nonexpansive mapping is strongly quasi-nonexpansive.

K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi [1] first introduced the class of L-hybrid mappings in Hilbert spaces. Let $T : H \to H$ be a mapping and $L \geq 0$ a nonnegative number. T is said L-hybrid, signified as $T \in \mathcal{H}_L$, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + L\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in H.$$

Notice that for particular choices of L we obtain several important classes of nonlinear mappings. In fact

- $\mathcal{H}_0$ is the class of the nonexpansive mappings;
- $\mathcal{H}_2$ is the class of the nonspreading mappings;
- $\mathcal{H}_1$ is the class of the hybrid mappings.

Further, by the quasi-nonexpansivity of $L-$hybrid mappings, it follows that the assumption on S which is strongly quasi-nonexpansive mapping is weaker than the hypothesis averaged type nonspreading (or also $L-$hybrid). Of course, one can ask if some important $L-$hybrid mappings, as a nonspreading mapping, or a nonexpansive mapping, are already strongly quasi-nonexpansive. This is not always true, as shown in the following example.

Example 1. There exist nonexpansive mappings that are not strongly quasi-nonexpansive. Moreover, there exist nonspreading mappings that are not strongly quasi-nonexpansive. Let $T : H \to H$ be such that $Tx = -x$. Then T is nonexpansive but not strongly quasi-nonexpansive.

Moreover let $X = A \cup B \cup C \subset H$, where

$$A = \{x \in H : \|x\| \leq 1\}; \quad B = \{x \in H : 1 < \|x\| < 2\};$$
$$C = \{x \in H : 2 \leq \|x\| \leq 3\}.$$

Define $S : X \to X$ by

$$Sx := \begin{cases} x, & \text{if } x \in A, \\ x/\|x\|, & \text{if } x \in B, \\ 0, & \text{if } x \in C. \end{cases} \tag{2.1}$$

One can see that S is a nonspreading mapping, distinguishing three cases ($x \in A$, $y \in B$), ($x \in A$, $y \in C$), ($x \in B$, $y \in C$).

To see that S is not strongly quasi-nonexpansive take x_0 with $\|x_0\| = 1$. Then $x_0 \in Fix(S)$. Moreover, define $z_n = \left(2 + \frac{1}{n}\right) x_0$. Then $Sz_n = 0$ and

$$\|z_n - x_0\| - \|Sz_n - x_0\| = 1 + \frac{1}{n} - 1 \to 0,$$

but $Sz_n - z_n = (2 + \frac{1}{n})x_0 \to 2x_0$.

Conversely, we have the

Example 2. There exist strongly quasi-nonexpansive mappings that are not $L-$hybrid mappings for any L (and hence that are not ever type average S_δ with S $L-$hybrid). Let $H = \mathbb{R}$. Define

$$T(0) = 0,$$
$$T(n) = -n + \frac{1}{n+1}, \quad T(-n) = n - \frac{1}{n+1}, \quad \forall n \in \mathbb{N}, n > 0.$$

Then define in linear way T on each interval $[n, n+1]$, $n \in \mathbb{Z}$. One can see easily that $Fix(T) = 0$ and T is strongly quasi-nonexpansive. Moreover, from the fact that for large $n \in \mathbb{Z}$, T is defined *almost* as $-I$, one can prove that there can not be $L-$hybrid for any $L \geq 0$.

The solid bases on which our proof rest are given by the following lemmas:

Lemma 1 [Xu's Lemma]. *[18] Assume $(a_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - s_n)a_n + s_n\sigma_n + \gamma_n, \quad n \geq 0,$$

where $(s_n)_n$ is a sequence in $[0, 1]$ and $(\sigma_n)_n$ is a sequence in $\mathbb{R}$ such that,

$$1.\ \sum_{n=1}^{\infty} s_n = \infty; \quad 2.\ \limsup_{n\to\infty} \sigma_n \leq 0; \quad 3.\ \gamma_n \geq 0, \quad \sum_{n=1}^{\infty} \gamma_n < \infty;$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 [Maingé's Lemma]. *[11] Let $(\gamma_n)_n$ be real sequence that has a subsequence (γ_{n_j}) which satisfies $\gamma_{n_j} < \gamma_{n_j+1}$ for all j. Then there exists an increasing sequence of integers $(\tau(n))_{n\geq n_0}$ sastisfying:*

$$1.\ \lim_n \tau(n) = +\infty; \quad 2.\ \gamma_{\tau(n)} \leq \gamma_{\tau(n)+1}, \quad \textit{for all}\ n \geq n_0;$$
$$3.\ \gamma_n \leq \gamma_{\tau(n)+1}, \quad \textit{for all}\ n \geq n_0.$$

For the sake of completeness we recall the definition of demiclosedness.

Definition 1. [14] Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping such that $Fix(T) \neq \emptyset$. The mapping $I - T$ is said demiclosed at 0 if for every sequence $(x_n)_{n\in N}$ weakly convergent to $p \in H$ such that $x_n - Tx_n \to 0$, it follows that $p \in Fix(T)$.

To prove the main result, our reasoning is inspired by the ideas contained in [7,12,17].

Theorem 3. *Let H be a Hilbert space and let C be a closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be a strongly quasi-nonexpansive mapping such that $I - S$ is demiclosed in 0. Assume that $Fix(T) \cap Fix(S) \neq \emptyset$. Let u be a fixed anchor in C. If $(\alpha_n)_n, (\beta_n)_n$ are sequences in $[0,1]$, we define a sequence $(x_n)_{n \in N}$*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n T x_n + (1-\beta_n) S x_n], \quad n \geq 1. \end{cases} \tag{2.2}$$

Then

(1) If $\sum_n (1-\beta_n) < \infty$, $\alpha_n \to 0$, $\sum_n \alpha_n = \infty$, $\sum_n |\alpha_n - \alpha_{n+1}| < \infty$, then (x_n) strongly converges to $\bar{p} \in Fix(T)$ that is the unique point in $Fix(T)$ that solves the variational inequality

$$\langle \bar{p} - u, x - \bar{p} \rangle \geq 0, \qquad \forall x \in Fix(T), \tag{2.3}$$

i.e. $\bar{p} = P_{Fix(T)} u$.

(2) If $\sum_n \beta_n < \infty$, $\alpha_n \to 0$, $\sum_n \alpha_n = +\infty$, $\frac{\beta_n}{\alpha_n} \to 0$, then $(x_n)_n$ converges strongly to $\tilde{p} \in Fix(S)$ that is the unique solution in $Fix(S)$ of the variational inequality

$$\langle \tilde{p} - u, x - \tilde{p} \rangle \geq 0, \qquad \forall x \in Fix(S), \tag{2.4}$$

i.e. $\tilde{p} = P_{Fix(S)} u$.

(3) If $\liminf_n \beta_n (1-\beta_n) > 0$, $\alpha_n \to 0$, $\sum_n \alpha_n = +\infty$, then (x_n) strongly converges to $p_0 \in Fix(T) \cap Fix(S)$ is the unique solution in $Fix(T) \cap Fix(S)$ of the variational inequality

$$\langle p_0 - u, x - p_0 \rangle \geq 0, \qquad \forall x \in Fix(T) \cap Fix(S), \tag{2.5}$$

i.e. $p_0 = P_{Fix(T) \cap Fix(S)} u$.

Proof. In the sequel, we denote by $O(1)$ any bounded real sequence (so, for example, $O(1)+O(1) = O(1)$). First of all, we check that $(x_n)_{n \in N}$ is a bounded sequence. Indeed, let $(U_n)_{n \in N}$ a sequence defined by $U_n = \beta_n T + (1-\beta_n) S$ and $z \in Fix(T) \cap Fix(S)$. Then,

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n u + (1-\alpha_n) U_n x_n - z\| \\ &= \|(1-\alpha_n)(U_n x_n - z) + \alpha_n (u - z)\| \\ &\leq (1-\alpha_n)\|x_n - z\| + \alpha_n \|u - z\| \\ \text{(by convexity)} &\leq \max\{\|x_n - z\|, \|u - z\|\} \\ \text{(from induction)} &\leq \max\{\|x_1 - z\|, \|u - z\|\}. \end{aligned}$$

Then, (x_n) is bounded.

Moreover

$$x_{n+1} - U_n x_n = \alpha_n(u - U_n x_n) \to 0, \qquad n \to \infty, \tag{2.6}$$

since $\alpha_n \to 0$.

Proof. (1) The the key will be to prove that $x_{n+1} - x_n \to 0$. In order to show this, we calculate

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1-\alpha_n)(U_n x_n - U_{n-1}x_{n-1}) \\
&- (\alpha_n - \alpha_{n-1})U_{n-1}x_{n-1} + (\alpha_n - \alpha_{n-1})u\| \\
&= \|(1-\alpha_n)(U_n x_n - U_{n-1}x_{n-1}) + (\alpha_{n-1} - \alpha_n)(U_{n-1}x_{n-1} - u)\| \\
&= \|(\alpha_{n-1} - \alpha_n)(U_{n-1}x_{n-1} - u) \\
&+ (1-\alpha_n)[\beta_n T x_n + (1-\beta_n)Sx_n - \beta_{n-1}Tx_{n-1} \\
&- (1-\beta_{n-1})Sx_{n-1}]\| \\
&= \|(\alpha_{n-1} - \alpha_n)(U_{n-1}x_{n-1} - u) \\
&+ (1-\alpha_n)[\beta_n(Tx_n - Tx_{n-1}) + (1-\beta_n)(Sx_n - Sx_{n-1}) \\
&+ (\beta_n - \beta_{n-1})Tx_{n-1} + (\beta_{n-1} - \beta_n)Sx_{n-1}]\| \\
(T \text{ nonexp.}) &\le |\alpha_{n-1} - \alpha_n| O(1) \\
&+ (1-\alpha_n)[\beta_n\|x_n - x_{n-1}\| + (1-\beta_n)O(1) + |\beta_n - \beta_{n-1}|O(1)] \\
&= (1 - s_n)\|x_n - x_{n-1}\| + \gamma_n,
\end{aligned}$$

where $s_n = 1 - \beta_n + \alpha_n\beta_n \ge \alpha_n O(1)$ eventually, $\gamma_n = |\alpha_{n-1} - \alpha_n|O(1) + [(1-\alpha_n)(1-\beta_n) + |\beta_n - \beta_{n-1}|]O(1)$.

Thanks to hypotheses on α_n, β_n we see that $s_n \to 0$, $\sum_n s_n = +\infty$ and $\sum_n \gamma_n < \infty$. This is sufficient, from Xu's Lemma, to conclude $x_{n+1} - x_n \to 0$. From this and (2.6) follows immediately

$$x_n - U_n x_n \to 0, \tag{2.7}$$

since $x_n - U_n x_n = (x_n - x_{n+1}) - (x_{n+1} - U_n x_n)$. Moreover

$$\begin{aligned}
\|x_n - U_n x_n\| &= \|x_n - \beta_n T x_n - (1-\beta_n)Sx_n\| \\
&\ge \|x_n - \beta_n T x_n\| - (1-\beta_n)\|Sx_n\|, \\
\|x_n - \beta_n T x_n\| &\le \|x_n - U_n x_n\| + (1-\beta_n)O(1),
\end{aligned}$$

and so from (2.7) and hypotheses $\sum_n (1-\beta_n) < \infty$, we have also $x_n - \beta_n T x_n \to 0$.

From this we deduce also $x_n - Tx_n \to 0$, and this gives that any weak limit of (x_n) is in $Fix(T)$, since T is nonexpansive, and thus the Principle of Demiclosedness is satisfied.

Now we can show that $x_n \to \bar{p}$, where $\bar{p}$ is the unique solution in $Fix(T)$ of the variational inequality (2.3). We show first that

$$\limsup_n \langle U_n x_n - \bar{p}, u - \bar{p}\rangle \le 0. \tag{2.8}$$

Indeed, let (x_{n_k}) be a subsequence of (x_n) such that

$$\limsup_n \langle x_n - \bar{p}, u - \bar{p}\rangle = \lim_k \langle x_{n_k} - \bar{p}, u - \bar{p}\rangle \tag{2.9}$$

and $x_{n_k} \rightharpoonup z$. Then $z \in Fix(T)$ and so, from (2.9)

$$\limsup_n \langle x_n - \bar{p}, u - \bar{p}\rangle = \langle z - \bar{p}, u - \bar{p}\rangle$$

and this is nonpositive by definition of $\bar{p}$.

We have

$$\begin{aligned}
\limsup_{n\to\infty}\langle U_n x_n - \bar{p}, u - \bar{p}\rangle &= \limsup_{n\to\infty}[\langle x_n - \bar{p}, u - \bar{p}\rangle + \langle U_n x_n - x_n, u - \bar{p}\rangle] \\
\text{(from (2.7))} &= \limsup_{n\to\infty}\langle x_n - \bar{p}, u - \bar{p}\rangle \\
\text{(by (2.9))} &\leq 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|x_{n+1} - \bar{p}\| &= \|(1-\alpha_n)(U_n x_n - \bar{p}) + \alpha_n(u - \bar{p})\|^2 \\
&= (1-\alpha_n)^2\|U_n x_n - \bar{p}\|^2 + \alpha_n^2\|u - \bar{p}\|^2 \\
&+ 2\alpha_n\langle(1-\alpha_n)(U_n x_n - \bar{p}), u - \bar{p}\rangle \\
&= (1-\alpha_n)^2\|\beta_n(Tx_n - \bar{p}) + (1-\beta_n)(Sx_n - \bar{p})\|^2 \\
&+ \alpha_n^2\|u - \bar{p}\|^2 + 2\alpha_n\langle U_n x_n - \bar{p}, u - \bar{p}\rangle \\
&- 2\alpha_n^2\langle U_n x_n - \bar{p}, u - \bar{p}\rangle \\
\text{(by } T \text{ quasi-nonexp.)} &\leq (1-\alpha_n)^2[\beta_n\|x_n - \bar{p}\| + (1-\beta_n)O(1)]^2 + \alpha_n^2 O(1) \\
&+ 2\alpha_n\langle U_n x_n - \bar{p}, u - \bar{p}\rangle + 2\alpha_n^2 O(1) \\
&\leq (1-\alpha_n)^2\|x_n - \bar{p}\|^2 + (1-\beta_n)O(1) + \alpha_n^2 O(1) \\
&+ 2\alpha_n\langle U_n x_n - \bar{p}, u - \bar{p}\rangle \\
&\leq (1-\alpha_n)^2\|x_n - \bar{p}\|^2 + (1-\beta_n)O(1) + \alpha_n^2 O(1) \\
&+ 2\alpha_n\langle U_n x_n - \bar{p}, u - \bar{p}\rangle \\
&= (1-2\alpha_n)\|x_n - \bar{p}\|^2 + (1-\beta_n)O(1) + \alpha_n^2 O(1) \\
&+ 2\alpha_n\langle U_n x_n - \bar{p}, u - \bar{p}\rangle \\
&= (1-s_n)\|x_n - \bar{p}\|^2 + s_n\sigma_n + \gamma_n,
\end{aligned}$$

where

$$s_n = 2\alpha_n, \quad \sigma_n = \alpha_n O(1) + 2\langle U_n x_n - \bar{p}, u - \bar{p}\rangle, \quad \gamma_n = (1-\beta_n)O(1).$$

Thanks to the hypotheses on α_n, β_n and (2.8), from Xu's Lemma again, we obtain $x_n \to \bar{p}$. □

Proof. (2) Let $\tilde{p}$ be the unique solution of variational inequality (2.4). We want to show that $x_n \to \tilde{p}$. We compute,

$$\begin{array}{rcl} \|x_{n+1}-\tilde{p}\|^2 & = & \|\alpha_n u + (1-\alpha_n)U_n x_n - \tilde{p} + \alpha_n\tilde{p} - \alpha_n\tilde{p}\|^2 \\ & = & \|\alpha_n(u-\tilde{p}) + (1-\alpha_n)(U_n x_n - \tilde{p})\|^2 \\ & \leq & (1-\alpha_n)^2\|U_n x_n - \tilde{p}\|^2 + 2\alpha_n\langle u-\tilde{p}, x_{n+1}-\tilde{p}\rangle \\ & = & (1-\alpha_n)^2\|\beta_n(Tx_n-\tilde{p}) + (1-\beta_n)(Sx_n-\tilde{p})\|^2 \\ & + & 2\alpha_n\langle u-\tilde{p}, x_{n+1}-\tilde{p}\rangle \\ \text{(by } S \text{ quasi-nonexp.)} & \leq & (1-\alpha_n)^2\|x_n-\tilde{p}\|^2 \\ & + & \beta_n O(1) + 2\alpha_n\langle x_{n+1}-\tilde{p}, u-\tilde{p}\rangle. \end{array} \tag{2.10}$$

At this point we distinguish two cases: or the sequence $\|x_n - \tilde{p}\|$ is eventually not increasing or no.

Alternative 1. $(\|x_n - \tilde{p}\|)$ is eventually not increasing, so $\|x_{n+1} - \tilde{p}\| \leq \|x_n - \tilde{p}\|$, $\forall n \geq N$.

Putting $\sigma_n = \langle x_{n+1}-\tilde{p}, u-\tilde{p}\rangle$, $\gamma_n = \beta_n O(1)$ and since $(1-\alpha_n)^2 \leq (1-\alpha_n)$, we can rewrite (2.10) as $\|x_{n+1}-\tilde{p}\|^2 \leq (1-\alpha_n)\|x_n-\tilde{p}\|^2 + \alpha_n\sigma_n + \gamma_n$, so the thesis $x_n \to \tilde{p}$ will follows again by Xu's Lemma if we are able to show that

$$\limsup_n \langle x_{n+1}-\tilde{p}, u-\tilde{p}\rangle \leq 0. \tag{2.11}$$

Note that until now we have not used the hypothesis of strong quasi-nonexpansivity of S. Now, since $(\|x_n - \tilde{p}\|)$ is definitively not increasing, there exists the $\lim_n \|x_n - \tilde{p}\|$. Then

$$\begin{array}{rcl} 0 & = & \lim_{n\to\infty}(\|x_{n+1}-\tilde{p}\| - \|x_n-\tilde{p}\|) \\ & \leq & \liminf_{n\to\infty}(\alpha_n\|u-\tilde{p}\| + (1-\alpha_n)\|U_n x_n-\tilde{p}\| - \|x_n-\tilde{p}\|) \\ \text{(by } \alpha_n \to 0) & = & \liminf_{n\to\infty}(\|U_n x_n-\tilde{p}\| - \|x_n-\tilde{p}\|) \\ & = & \liminf_{n\to\infty}(\|\beta_n(Tx_n-\tilde{p}) + (1-\beta_n)(Sx_n-\tilde{p})\| - \|x_n-\tilde{p}\|) \\ \text{(by } \beta_n \to 0) & = & \liminf_{n\to\infty}(\|Sx_n-\tilde{p}\| - \|x_n-\tilde{p}\|) \\ (S \text{ quasi-nonexp. }) & \leq & \limsup_{n\to\infty}(\|x_n-\tilde{p}\| - \|x_n-\tilde{p}\|) = 0. \end{array}$$

Thus

$$\lim_n(\|Sx_n-\tilde{p}\| - \|x_n-\tilde{p}\|) = 0.$$

From the strong quasi-nonexpansivity of S, we deduce

$$Sx_n - x_n \to 0. \tag{2.12}$$

At this point, by using the demiclosedness of $(I-S)$ in 0 we can proceed as in the Proof of (1) to show (2.11). The statement is proved when the Alternative 1 holds.

Alternative 2. $(\|x_n - \tilde{p}\|)$ is not definitively not increasing, i.e. there exists a subsequence $(\|x_{n_j} - \tilde{p}\|)$ such that $\|x_{n_j} - \tilde{p}\| < \|x_{n_j+1} - \tilde{p}\|$, $\forall j \in \mathbb{N}$.

From Maingé's Lemma it follows that there exists an increasing sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ satisfying

$$\lim_n \tau(n) = +\infty, \quad \|x_{\tau(n)} - \tilde{p}\| \leq \|x_{\tau(n)+1} - \tilde{p}\|, \tag{2.13}$$

$$\|x_n - \tilde{p}\| \leq \|x_{\tau(n)+1} - \tilde{p}\|, \quad \forall n \geq n_0. \tag{2.14}$$

Then

$$0 \leq \liminf_n (\|x_{\tau(n)+1} - \tilde{p}\| - \|x_{\tau(n)} - \tilde{p}\|).$$

Retracing the same inequalities used to obtain (2.12) with $\tau(n)$ instead of n, we obtain

$$\lim_n \|Sx_{\tau(n)} - \tilde{p}\| - \|x_{\tau(n)} - \tilde{p}\| = 0.$$

Again the strong quasi-nonexpansivity yields

$$Sx_{\tau(n)} - x_{\tau(n)} \to 0 \tag{2.15}$$

and from the demiclosedness of $I - S$ in 0, we deduce as above

$$\limsup_n \langle x_{\tau(n)+1} - \tilde{p}, u - \tilde{p}\rangle \leq 0. \tag{2.16}$$

Incidentally, we observe that

$$\begin{aligned} \|x_{\tau(n)+1} - x_{\tau(n)}\| &= \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + (1-\alpha_{\tau(n)})\beta_{\tau(n)}O(1) \\ &+ (1-\alpha_{\tau(n)})\|Sx_{\tau(n)} - x_{\tau(n)}\|, \end{aligned}$$

and so, from (2.15) it follows also $x_{\tau(n)+1} - x_{\tau(n)} \to 0$. We replace in (2.10) n with $\tau(n)$ and we get

$$\begin{aligned} \|x_{\tau(n)+1} - \tilde{p}\| &\leq (1-\alpha_{\tau(n)})^2\|x_{\tau(n)} - \tilde{p}\|^2 \\ &+ 2\alpha_{\tau(n)}\langle x_{\tau(n)+1} - \tilde{p}, u - \tilde{p}\rangle + \beta_{\tau(n)}O(1) \\ \text{(by property (2.13))} &\leq (1-\alpha_{\tau(n)})^2\|x_{\tau(n)+1} - \tilde{p}\|^2 \\ &+ 2\alpha_{\tau(n)}\langle x_{\tau(n)+1} - \tilde{p}, u - \tilde{p}\rangle + \beta_{\tau(n)}O(1), \end{aligned}$$

consequently

$$\begin{aligned} 2\alpha_{\tau(n)}\|x_{\tau(n)+1} - \tilde{p}\|^2 &\leq (\alpha_{\tau(n)})^2\|x_{\tau(n)+1} - \tilde{p}\|^2 \\ &+ 2\alpha_{\tau(n)}\langle x_{\tau(n)+1} - \tilde{p}, u - \tilde{p}\rangle + \beta_{\tau(n)}O(1) \end{aligned}$$

and dividing by $\alpha_{\tau(n)}$, we have

$$\begin{aligned} 0 &\leq 2\|x_{\tau(n)+1} - \tilde{p}\|^2 \leq \alpha_{\tau(n)}\|x_{\tau(n)+1} - \tilde{p}\|^2 \\ &+ 2\langle x_{\tau(n)+1} - \tilde{p}, u - \tilde{p}\rangle + \frac{\beta_{\tau(n)}}{\alpha_{\tau(n)}}O(1). \end{aligned}$$

Passing to limsup and recalling the hypothesis $\frac{\beta_n}{\alpha_n} \to 0$ and (2.16), we obtain $\lim_n \|x_{\tau(n)} - \tilde{p}\| \leq \lim_n \|x_{\tau(n)+1} - \tilde{p}\| = 0$. The (2.14) ensures that also $x_n \to \tilde{p}$. □

Proof. (3) Let p_0 the unique point in $Fix(S) \cap F$ that satisfies the variational inequality (2.5). Then

$$\begin{aligned} \|U_n x_n - p_0\|^2 &= \|\beta_n(Tx_n - p_0) + (1-\beta_n)(Sx_n - p_0)\|^2 \\ &= \beta_n \|Tx_n - p_0\|^2 + (1-\beta_n)\|Sx_n - p_0\|^2 - \beta_n(1-\beta_n)\|Tx_n - Sx_n\|^2 \\ &\le \|x_n - p_0\|^2 - \beta_n(1-\beta_n)\|Tx_n - Sx_n\|^2, \end{aligned} \tag{2.17}$$

$$\begin{aligned} \|x_{n+1} - p_0\|^2 &= \|\alpha_n(u - p_0) + (1-\alpha_n)(U_n x_n - p_0)\|^2 \\ &\le (1-\alpha_n)^2 \|U_n x_n - p_0\|^2 + \alpha_n^2 O(1) + \alpha_n O(1) \\ &\text{(by (2.17))} \ \le \|x_n - p_0\|^2 - \beta_n(1-\beta_n)\|Tx_n - Sx_n\|^2 + \alpha_n^2 O(1) \\ &+ \alpha_n O(1). \end{aligned} \tag{2.18}$$

Also now we distinguish two cases.

Alternative 1. $(\|x_n - p_0\|)$ is eventually not increasing, $\|x_{n+1} - p_0\| \le \|x_n - p_0\|, \forall n \ge N$.

Then there exists $\lim_n \|x_n - \tilde{p}\|$, so (2.18) furnish

$$\beta_n(1-\beta_n)\|Tx_n - Sx_n\| \le \|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2 + \alpha_n^2 O(1) + \alpha_n O(1)$$

and so, by hypothesis $\liminf_n \beta_n(1-\beta_n) > 0$, we deduce

$$Tx_n - Sx_n \to 0. \tag{2.19}$$

Moreover,

$$\begin{aligned} 0 &= \lim_{n\to\infty} (\|x_{n+1} - p_0\| - \|x_n - p_0\|) \\ &\le \liminf_{n\to\infty} (\alpha_n \|u - p_0\| + (1-\alpha_n)\|U_n x_n - p_0\| - \|x_n - p_0\|) \\ (\text{by} \alpha_n \to 0) &= \liminf_{n\to\infty} (\|U_n x_n - p_0\| - \|x_n - p_0\|) \\ &= \liminf_{n\to\infty} (\|\beta_n(Tx_n - p_0) + (1-\beta_n)(Sx_n - p_0)\| - \|x_n - p_0\|) \\ &\le \liminf_{n\to\infty} (\beta_n(\|Tx_n - p_0\| - \|x_n - p_0\|) \\ &+ (1-\beta_n)(\|Sx_n - p_0\| - \|x_n - p_0\|)) \\ &\le \limsup_{n\to\infty} (\beta_n(\|Tx_n - p_0\| - \|x_n - p_0\|) \\ &+ (1-\beta_n)(\|Sx_n - p_0\| - \|x_n - p_0\|)) \\ (\ T, S \text{ quasi-nonep.}) &\le 0. \end{aligned}$$

But then

$$\lim_n \{\beta_n[\|Tx_n - p_0\| - \|x_n - p_0\|] + (1-\beta_n)[\|Sx_n - p_0\| - \|x_n - p_0\|]\} = 0.$$

Since both the addends are non positive and the limit of the sum is zero, it follows that

$$\lim_n \beta_n[\|Tx_n - p_0\| - \|x_n - p_0\|] = \lim_n (1-\beta_n)[\|Sx_n - p_0\| - \|x_n - p_0\|] = 0. \tag{2.20}$$

But then the hypothesis $\liminf_n \beta_n(1-\beta_n) > 0$ implies

$$\lim_n \|Tx_n - p_0\| - \|x_n - p_0\| = \lim_n \|Sx_n - p_0\| - \|x_n - p_0\| = 0. \tag{2.21}$$

From strong quasi-nonexpansivity of S it follows

$$Sx_n - x_n \to 0. \tag{2.22}$$

Again $x_n - Tx_n = x_n - Sx_n + Sx_n - Tx_n$, so by (2.19) and (2.20),

$$x_n - Tx_n \to 0. \tag{2.23}$$

We show now that

$$\limsup_n \langle x_n - p_0, u - p_0\rangle \leq 0. \tag{2.24}$$

Indeed, select a subsequence $x_{n_k} \rightharpoonup z$ and such that

$$\limsup_n \langle x_n - p_0, u - p_0\rangle = \lim_k \langle x_{n_k} - p_0, u - p_0\rangle = \langle z - p_0, u - p_0\rangle.$$

But by the demiclosedness of both T and S and by (2.22) and (2.23), one deduces that $z \in Fix(S) \cap Fix(T)$, and so, by definition of p_0, (2.24) is obtained. Moreover, from (2.22) and (2.23) at once follows,

$$U_n x_n - x_n \to 0. \tag{2.25}$$

Finally we are able to show $x_n \to p_0$. Indeed,

$$\begin{aligned}
\|x_{n+1} - p_0\|^2 &= \|(1-\alpha_n)(U_n x_n - p_0) + \alpha_n(u - p_0)\|^2 \\
&= (1-\alpha_n)^2\|U_n x_n - p_0\|^2 + \alpha_n^2\|u - p_0\|^2 + 2\alpha_n\langle(1-\alpha_n)(U_n x_n - p_0), u - p_0\rangle \\
(\,U_n \text{ quasi-nonexp.}) &\leq (1-\alpha_n)^2\|x_n - p_0\|^2 + \alpha_n^2[\|u - p_0\|^2 - 2\langle U_n x_n - p_0, u - p_0\rangle] \\
&\quad + 2\alpha_n\langle U_n x_n - x_n, u - p_0\rangle + 2\alpha_n\langle x_n - p_0, u - p_0\rangle \\
&\leq (1-\alpha_n)\|x_n - p_0\|^2 + \alpha_n^2 O(1) + 2\alpha_n\langle U_n x_n - x_n, u - p_0\rangle \\
&\quad + 2\alpha_n\langle x_n - p_0, u - p_0\rangle.
\end{aligned}$$

Put

$$\begin{aligned}
s_n &= 1 - (1-\alpha_n)^2, \\
\sigma_n &= 2\langle U_n x_n - x_n, u - p_0\rangle + 2\langle x_n - p_0, u - p_0\rangle + \alpha_n O(1),
\end{aligned}$$

the thesis follows again by Xu's Lemma, taking account of (2.24) and (2.25).

Alternative 2. $(\|x_n - p_0\|)$ is not egventually not increasing, i.e. there exists a subsequence $(\|x_{n_j} - p_0\|)$ such that $\|x_{n_j} - p_0\| < \|x_{n_{j+1}} - p_0\|$, $\forall j \in \mathbb{N}$.

From Maingé's Lemma it follows that there exists an increasing sequence of integers $(\tau(n))_{n\in\mathbb{N}}$ satisfying (2.13) and (2.14). Then

$$\begin{aligned} 0 &\le \liminf_{n\to\infty}(\|x_{\tau(n)+1}-p_0\|-\|x_{\tau(n)}-p_0\|) \\ &\le \limsup_{n\to\infty}(\|x_{\tau(n)+1}-p_0\|-\|x_{\tau(n)}-p_0\|) \\ &\le \limsup_{n\to\infty}(\|x_{n+1}-p_0\|-\|x_n-p_0\|) \\ &\le \limsup_{n\to\infty}(\|(1-\alpha_n)(U_nx_n-p_0)+\alpha_n(u-p_0)\|-\|x_n-p_0\|) \\ \text{(by } U_n \text{ quasi-nonexp.)} &\le \limsup_{n\to\infty}((1-\alpha_n)\|x_n-p_0\|+\alpha_nO(1)-\|x_n-p_0\|) \\ \text{(by } \alpha_n\to 0) &= 0. \end{aligned}$$

Hence

$$\lim_n\left[\|x_{\tau(n)+1}-p_0\|-\|x_{\tau(n)}-p_0\|\right]=0. \tag{2.26}$$

Now, retracing the same inequalities used to obtain (2.22) with $\tau(n)$ instead of n, we have

$$Sx_{\tau(n)}-x_{\tau(n)}\to 0. \tag{2.27}$$

Moreover, we can rewrite (2.18) as

$$0\le\beta_{\tau(n)}(1-\beta_{\tau(n)})\|Tx_{\tau(n)}-Sx_{\tau(n)}\|^2\le\|x_{\tau(n)}-p_0\|^2-\|x_{\tau(n)+1}-p_0\|^2+\alpha_nO(1),$$

and so, from (2.26) and the hypothesis $\liminf_n\beta_n(1-\beta_n)>0$,

$$Tx_{\tau(n)}-Sx_{\tau(n)}\to 0. \tag{2.28}$$

Again,

$$x_{\tau(n)}-Tx_{\tau(n)}=x_{\tau(n)}-Sx_{\tau(n)}+Sx_{\tau(n)}-Tx_{\tau(n)},$$

so, by (2.27) and (2.28), $x_{\tau(n)}-Tx_{\tau(n)}\to 0$, so also

$$x_{\tau(n)}-U_{\tau(n)}x_{\tau(n)}\to 0. \tag{2.29}$$

The same reasoning used to have (2.24) can be now repeated with $\tau(n)$ instead of n obtaining $\limsup_n\langle x_{\tau(n)}-p_0,u-p_0\rangle\le 0$. Following the same line to prove (2.24), we replace n with $\tau(n)$ and we obtain

$$\limsup_{n\to\infty}\langle x_{\tau(n)}-p_0,u-p_0\rangle\le 0. \tag{2.30}$$

We compute,

$$\begin{aligned} \|x_{\tau(n)+1}-p_0\| &\le (1-\alpha_{\tau(n)})^2\|x_{\tau(n)}-p_0\|^2+\alpha_{\tau(n)}^2O(1) \\ &+ 2\alpha_{\tau(n)}\langle U_{\tau(n)}x_{\tau(n)}-x_{\tau(n)},u-p_0\rangle \\ &+ 2\alpha_{\tau(n)}\langle x_{\tau(n)}-p_0,u-p_0\rangle \\ \text{(by property (2.13))} &\le (1-\alpha_{\tau(n)})^2\|x_{\tau(n)+1}-p_0\|^2+\alpha_{\tau(n)}^2O(1) \\ &+ 2\alpha_{\tau(n)}\langle U_{\tau(n)}x_{\tau(n)}-x_{\tau(n)},u-p_0\rangle \\ &+ 2\alpha_{\tau(n)}\langle x_{\tau(n)}-p_0,u-p_0\rangle, \end{aligned}$$

consequently

$$\begin{aligned} 2\alpha_{\tau(n)}\|x_{\tau(n)+1}-p_0\|^2 &\leq \alpha^2_{\tau(n)}O(1)+2\alpha_{\tau(n)}\langle x_{\tau(n)}-p_0, u-p_0\rangle \\ &+ 2\alpha_{\tau(n)}\langle U_{\tau(n)}x_{\tau(n)}-x_{\tau(n)}, u-p_0\rangle, \end{aligned}$$

dividing by $\alpha_{\tau(n)}$, we get

$$\begin{aligned} 0 &\leq 2\|x_{\tau(n)+1}-p_0\|^2 \leq \alpha_{\tau(n)}O(1) \\ &+ 2\langle U_{\tau(n)}x_{\tau(n)}-x_{\tau(n)}, u-p_0\rangle + 2\langle x_{\tau(n)}-p_0, u-p_0\rangle. \end{aligned}$$

Taking the limsup and recalling the hypothesis (2.29) and (2.30), we obtain

$$\lim_{n\to\infty}\|x_{\tau(n)}-p_0\| \leq \lim_{n\to\infty}\|x_{\tau(n)+1}-p_0\| = 0.$$

Once again by (2.14) we deduce $x_n \to p_0$. □
□

Remark 2. We can show that the same thesis of Theorem 3 holds for averaged type mappings of two quasi-nonexpansive mappings, as a consequence of Remark 1 and using the following inequality established in [17]

$$\langle x - T_\delta x, x-p\rangle \geq \frac{\delta}{2}\|x-Tx\|^2, \quad \forall x \in C, p \in Fix(T) \tag{2.31}$$

instead of the (1/2)-inverse strong monotonicity of the mapping $I-T$ when T is a nonexpansive mapping. Precisely, if we replace in the scheme (2.2) the mappings T, S with the averaged type mappings T_δ, S_δ, we obtain the same thesis of Theorem 3 under the assumptions:

- $T, S : C \to C$ be two quasi-nonexpansive mappings such that $I-T, I-S$ are demiclosed at 0;
- $Fix(S) \cap Fix(T) \neq \emptyset$;
- same hypotheses on the coefficients $(\alpha_n)_{n\in N}$ and $(\beta_n)_{n\in N}$;

for a sequence (x_n) generated by the Algorithm

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)[\beta_n T_\delta x_n + (1-\beta_n)S_\delta x_n]. \end{cases} \tag{2.32}$$

It should be noted that the hypotheses of quasi-nonexpansivity and demiclosedness on the mappings involved in our algorithm (2.32) are independent (see Example 3).

Remark 3. In the literature, there exist some interesting mappings T which are quasi-nonexpansive and such that $I-T$ are demiclosed at 0. Let H be a real Hilbert space, let C be a nonempty, closed and convex subset of H and $T : C \to C$ such that $Fix(T) \neq \emptyset$. Next, we list some examples of such mappings.

1. T nonexpansive mapping, [3]; 2. T nonspreading mapping, [8];
3. T Chatterjea mapping, [16]; 4. T L-hybrid mapping, [8].

Further, there exist mappings T such that $I-T$ are demiclosed at 0 but not necessarily quasi-nonexpansive:

(a) [2] T continuous pseudocontractive mapping, i.e. if $\forall x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2;$$

(b) [4] T k-strictly pseudononspreading mapping, i.e. if there exists $k \in [0, 1)$ such that for all $x, y \in D(T)$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle.$$

Next, we will give an example of a quasi-nonexpansive mapping which does not satisfy the Demiclosedness Principle.

Example 3. Let

$$B^+_{\ell^2} = \{(x_i)_{i\in N} : \quad \sum_{i=1}^{\infty} x_i^2 \leq 1, \quad x_i \geq 0, i = 1, 2, ...\}$$

be the part of the unit ball of ℓ_2 contained in the positive cone.

Let $T : B^+_{\ell^2} \to B^+_{\ell^2}$ be defined by

$$Tx = \begin{cases} 0_{\ell^2}, & 0 \leq \|x\| \leq \frac{1}{2}, \\ \frac{x}{2\|x\|}, & \frac{1}{2} < \|x\| \leq 1. \end{cases}$$

We have $Fix(T) = \{0_{\ell^2}\}$. It is obvious that T is a quasi-nonexpansive mapping. It is easy to check that $I - T$ is not demiclosed at 0_{ℓ^2}.

3 An example

In this section, we illustrate our results with a numerical example.

Example 4. Let $T : B^+_{\ell^2} \to B^+_{\ell^2}$ be defined by $T(x) = (x_1 - x_1^3, x_2 - x_2^3, ...)$. Then, $Fix(T) = \{0_{\ell^2}\}$. We denote by $P_{B^+[0,\frac{1}{2}]}$ the metric projection from ℓ^2 onto

$$B^+[0, \frac{1}{2}] := \left\{ x = (x_i)_{i\in N} \in B^+_{\ell^2} : \quad \|x\| \leq \frac{1}{2} \right\}.$$

Let $S : B^+_{\ell^2} \to B^+_{\ell^2}$ defined by $S(x) = P_{B^+[0,\frac{1}{2}]}x$. It is obvious that $Fix(S) = B^+[0, \frac{1}{2}]$. Notice that $Fix(T) \cap Fix(S) = \{0_{l^2}\}$. T is a quasi-nonexpansive mapping, for all $x \in B^+_{l^2}$,

$$\|Tx - 0_{\ell^2}\|^2 = \sum_{n=1}^{\infty}(x_n - x_n^3)^2 = \sum_{n=1}^{\infty} x_n^2(1 - x_n^2)^2 \leq \|x - 0_{\ell^2}\|^2.$$

To show that T is a pseudo-contractive mapping we will see that $I - T = A$ is accretive, i.e. for all $x, y \in B^+_{\ell^2}$

$$\langle Ax - Ay, x - y\rangle \geq 0.$$

If $x \in B^+_{\ell^2}$, we denote by $x^3 = (x_1^3, x_2^3, \dots) \in B^+_{\ell^2}$. For all $x, y \in B^+_{\ell^2}$, we claim that

$$\langle x - y, x^3 - y^3\rangle \geq 0.$$

Indeed, notice that

$$\langle x-y, x^3-y^3\rangle = \sum_{i=1}^{\infty}(x_i-y_i)(x_i^3-y_i^3) \tag{3.1}$$

is a series of positive terms, since $(x_i-y_i)(x_i^3-y_i^3) \geq 0$ for every $i \in N$. It is easy to check that T is continuous. Therefore, from [2], $I-T$ is demiclosed at 0_{ℓ_2}.

On the other hand, T is not a nonexpansive mapping. Indeed, for $x = (1,0,\dots), y = (\frac{3}{4},0,\dots) \in B^+_{l^2}$, one has

$$\|Tx-Ty\| = \frac{3}{4} - \left(\frac{3}{4}\right)^3 = \frac{21}{64} > \|x-y\| = \frac{1}{4}.$$

It is well known that S is a nonexpansive mapping, and hence $I-S$ is demiclosed at 0 (see [3]).

We recall that $x = (x_1, x_2, ...) \in B^+_{\ell^2}$ and for $k \in \{2,3\}$, $x^k = (x_1^k, x_2^k, \dots) \in B^+_{\ell^2}$. Let $T_\delta(x) = x - \delta x^3$ and $S_\delta(x) = (1-\delta)x + \delta P_{B^+[0,\frac{1}{2}]}x$.

Notice that, if $x \in B^+[0,\frac{1}{2}]$, we get $S_\delta x = x$.

In the sequel we will consider two cases:

- in the first one we choose x_1, u in $\left\{x = (x_i)_{i\in N} \in B^+_{\ell^2} : \quad \|x\| > \frac{1}{2}\right\}$;
- in the second case we set $x_1, u \in B^+[0,\frac{1}{2}]$.

Case 1.

(i) We set $(\alpha_n)_{n\in N} = \left(\frac{1}{n+1}\right)_{n\in N}$, $(\beta_n)_{n\in N} = \left(1-\frac{1}{(n+1)^2}\right)_{n\in N}$, $x_1 = e_1$, $u = e_2$ and $\delta = \frac{1}{3}$.

We consider the sequence $(x_{n+1})_{n\in N}$ where $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ and

$$\begin{aligned} x_{n+1} &= \tfrac{1}{n+1}e_2 + \left(1-\tfrac{1}{n+1}\right) \\ &\times\left[\left(1-\tfrac{1}{(n+1)^2}\right)\left(x_n - \tfrac{1}{3}{x_n}^3\right) + \tfrac{1}{(n+1)^2}\left(\tfrac{2}{3}x_n + \tfrac{1}{3}P_{B^+[0,\frac{1}{2}]}x_n\right)\right]. \end{aligned}$$

First, we calculate

$$x_1 = (1,0,0,...); \quad x_2 = (3.541666666666666\cdot 10^{-1}, 5\cdot 10^{-1}, 0, ...);$$
$$x_3 = (2.257270516465794\cdot 10^{-1}, 6.397040059395266\cdot 10^{-1}, 0, ...);$$
$$\dots,\ x_{10^7} = (6.363036990406555\cdot 10^{-8}, 6.684363438458816\cdot 10^{-3}, 0, ...);\ \dots.$$

Next, we compute the norm of x_i, for $i = 1,2,3,\dots,10^7$:

$$\|x_1\| = 1; \quad \|x_2\| = 6.127267154105309\cdot 10^{-1};$$
$$\|x_3\| = 6.783611995538479\cdot 10^{-1}; \quad \|x_4\| = 6.860980019499106\cdot 10^{-1};$$
$$\|x_5\| = 6.681028956903493\cdot 10^{-1}; \quad \|x_6\| = 6.445625797917982\cdot 10^{-1};$$
$$\dots, \quad \|x_{10}\| = 5.664753569100269\cdot 10^{-1}; \quad \dots$$
$$\|x_{100}\| = 2.886513537288554\cdot 10^{-1};\ \dots,\ \|x_{10^7}\| = 6.684363438761673\cdot 10^{-3};$$

From Remark 2, we know that this algorithm converges to $\overline{p} = \{0_{\ell^2}\} = = P_{Fix(T)}u$.

(ii) We set $(\alpha_n)_{n\in N} = \left(\frac{1}{n+1}\right)_{n\in N}$, $(\beta_n)_{n\in N} = \left(\frac{1}{8^n}\right)_{n\in N}$, $x_1 = e_1$, $u = e_2$ and $\delta = \frac{1}{3}$.

We consider the sequence $(x_{n+1})_{n\in N}$ where $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ and

$$\begin{aligned} x_{n+1} &= \frac{1}{n+1}e_2 \\ &+ \left(1 - \frac{1}{n+1}\right)\left[\frac{1}{8^n}\left(x_n - \frac{1}{3}{x_n}^3\right) + \left(1 - \frac{1}{8^n}\right)\left(\frac{2}{3}x_n + \frac{1}{3}P_{B^+[0,\frac{1}{2}]}x_n\right)\right]. \end{aligned}$$

From Remark 2, we know that this algorithm converges to

$$\widehat{p} = P_{Fix(S)}u = (0, \frac{1}{2}, 0, ...).$$

(iii) We set $(\alpha_n)_{n\in N} = \left(\frac{1}{n+1}\right)_{n\in N}$, $(\beta_n)_{n\in N} = \left(\frac{9n}{10n+1}\right)_{n\in N}$, $x_1 = e_1$, $u = e_2$ and $\delta = \frac{1}{3}$. We consider the sequence $(x_{n+1})_{n\in N}$ where $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ and

$$\begin{aligned} x_{n+1} &= \frac{1}{n+1}e_2 + \left(1 - \frac{1}{n+1}\right) \\ &\times\left[\left(\frac{9n}{10n+1}\right)\left(x_n - \frac{1}{3}{x_n}^3\right) + \left(\frac{n+1}{10n+1}\right)\left(\frac{2}{3}x_n + \frac{1}{3}P_{B^+[0,\frac{1}{2}]}x_n\right)\right]. \end{aligned}$$

From Remark 2, we know that this algorithm converges to $p_0 = \{0_{\ell^2}\} = P_{Fix(T)\cap Fix(S)}u$.

Case 2.

If we choose $x_1, u \in B^+[0, \frac{1}{2}]$, let $(\alpha_n)_{n\in N}$ be a sequence in $(0, 1)$ and let $(\beta_n)_{n\in N}$ be a sequence in $[0, 1]$, we derive by induction that

$$0 \le \|x_n\| \le \frac{1}{2}, \quad \forall n \in N. \tag{3.2}$$

For $n = 1$, we have $\|x_1\| \le \frac{1}{2}$. We assume that(3.2) is true for some $k \in N$ and we prove that (3.2) is obtained for some $k + 1 \in N$.

$$\begin{aligned} &\|x_{k+1}\| \\ &= \left\|\alpha_k u + (1-\alpha_k)\left[\beta_k\left(x_k - \frac{1}{3}x_k^3\right) + (1-\beta_k)\left(\frac{2}{3}x_k + \frac{1}{3}Proj_{B[0,\frac{1}{2}]}x_k\right)\right]\right\| \\ &\le \alpha_k\|u\| + (1-\alpha_k)\left\|\beta_k x_k\left(1 - \frac{1}{3}x_k^2\right) + (1-\beta_k)x_k\right\| \\ &\le \alpha_k\frac{1}{2} + (1-\alpha_k)\left[\beta_k\left\|x_k\left(1 - \frac{1}{3}x_k^2\right)\right\| + (1-\beta_k)\|x_k\|\right] \\ &\le \alpha_k\frac{1}{2} + (1-\alpha_k)\left[\beta_k\frac{1}{2} + (1-\beta_k)\frac{1}{2}\right] = \frac{1}{2}. \end{aligned}$$

Moreover, $S_\delta x_n = x_n$, for all $n \in N$. Hence, the algorithm (1.4) becomes,

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1-\alpha_n)[\beta_n(x_n - \frac{1}{3}x_n^3) + (1-\beta_n)x_n] \\ &= \alpha_n u + (1-\alpha_n)[x_n - \frac{1}{3}\beta_n x_n^3], \quad \forall n \in N. \end{aligned}$$

We fix $u = (0, \frac{1}{4}, 0, ...)$ and $x_1 = (\frac{1}{2}, 0, 0, ...)$ in the following three cases.

(i) We set $(\alpha_n)_{n\in N} = \left(\frac{1}{n+1}\right)_{n\in N}$ and $(\beta_n)_{n\in N} = \left(1 - \frac{1}{(n+1)^2}\right)_{n\in N}$. We consider the sequence $(x_{n+1})_{n\in N}$ where $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ and

$$x_{n+1} = \frac{1}{n+1}u + \left(1 - \frac{1}{n+1}\right)\left[x_n - \frac{1}{3}\left(1 - \frac{1}{(n+1)^2}\right)x_n^3\right]. \tag{3.3}$$

From Remark 2, we know that the algorithm (3.3) converges to $\overline{p} = \{0_{\ell^2}\} = P_{Fix(T)}u$.

(ii) We set $(\alpha_n)_{n\in N} = \left(\frac{1}{n+1}\right)_{n\in N}$ and $(\beta_n)_{n\in N} = \left(\frac{1}{8^n}\right)_{n\in N}$.

We consider the sequence $(x_{n+1})_{n\in N}$ where $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ and

$$x_{n+1} = \frac{1}{n+1}u + \left(1 - \frac{1}{n+1}\right)\left[x_n - \frac{1}{3}\left(\frac{1}{8^n}\right)x_n^3\right]. \tag{3.4}$$

From Remark 2, we know that the algorithm (3.4) converges to $\widehat{p} = P_{Fix(S)}u = u = (0, \frac{1}{4}, 0, 0, ...)$.

(iii) We set $(\alpha_n)_{n\in N} = \left(\frac{1}{n+1}\right)_{n\in N}$ and $(\beta_n)_{n\in N} = \left(\frac{9n}{10n+1}\right)_{n\in N}$. We consider the sequence $(x_{n+1})_{n\in N}$ where $x_{n+1} = (x_1^{n+1}, x_2^{n+1}, ...)$ and

$$x_{n+1} = \frac{1}{n+1}u + \left(1 - \frac{1}{n+1}\right)\left[x_n - \frac{1}{3}\left(\frac{9n}{10n+1}\right)x_n^3\right]. \tag{3.5}$$

From Remark 2, we know that the algorithm (3.5) converges to $p_0 = \{0_{\ell^2}\} = P_{Fix(T)\cap Fix(S)}u$.

Remark 4. In the example in Case 1 and Case 2 we show that choosing different coefficients β_n our algorithm converges to the fixed point of T in (i), to the fixed point of S in (ii) and to common fixed point of two mappings in (iii).

In the first case, using Matlab Code we obtain that the speed of convergence of our iterative scheme to $\{0_{\ell^2}\}$ in (i) is faster than the one in (iii).

Moreover, in (ii) the sequence goes fast to $\widehat{p} = P_{Fix(S)}u = (0, \frac{1}{2}, 0, ...)$. If we compare the Case 1 and Case 2 of the previous example, we observe that in (iii) the sequence (x_n) converges to $\{0_{\ell^2}\}$ in Case 2 faster than in Case 1.

Next, we give the Matlab code for the example 4, (i).

```
format long e
n = 10000000;
x = zeros(n, 2);
x(1, :) = [1, 0];
```

$E = ones(n, 1);$
$u = [0, 1];$
$for \quad i = 1 : n$

$$\begin{aligned} x(i+1,:) &= \left(\frac{1}{1+i}\right) * u \\ &+ \left(1 - \frac{1}{i+1}\right) * \left(\left(1 - \frac{1}{(i+1)^2}\right) * \left(x(i,:) - \frac{1}{3} * x(i,:).^3\right) \right. \\ &+ \left. \frac{1}{(1+i)^2} * \left(\frac{2}{3} * x(i,:) + \frac{x(i,:)}{6 * E(i)}\right) \right); \end{aligned}$$

$E(i+1) = norm(x(i+1,:), 2);$
end
disp(' ')
disp('sequences')
disp(x)
disp('norm vector')
disp(E)

4 Conclusions

In this paper, we proposed an algorithm which converges to different fixed points, under suitable assumptions on the coefficients involved in the scheme. Moreover, MATLAB programming language has been used to obtain the computational results presented in the example to illustrate the iterative scheme.

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