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Fuzzy-Prešić-Ćirić Operators and Applications to Certain Nonlinear Differential Equations

Satish Shukla[a], Dhananjay Gopal[b] and Rosana Rodríguez-López[c]

[a] *Department of Applied Mathematics,*
Shri Vaishnav Institute of Technology & Science
Gram Baroli, Sanwer Road, 453331 Indore, India
[b] *Department of Applied Mathematics and Humanities,*
S.V. National Institute of Technology
Surat, 395-007 Gujarat, India
[c] *Departamento de Análisis Matemático, Facultad de Matemáticas,*
Universidad de Santiago de Compostela
15782 Santiago de Compostela, Spain
E-mail(*corresp.*): `rosana.rodriguez.lopez@usc.es`
E-mail: `satishmathematics@yahoo.co.in`
E-mail: `gopaldhananjay@yahoo.in`

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Abstract. In this paper, we introduce a new class of operators called fuzzy-Prešić-Ćirić operators. For this type of operators, the existence and uniqueness of fixed point in M-complete fuzzy metric spaces endowed with H-type t-norms are established. The results proved here generalize and extend some comparable results in the existing literature. An example is included which illustrates the main result of this paper. Moreover, some applications of our main theorem to the study of certain types of nonlinear differential equations are provided.

Keywords: Fuzzy-Prešić-Ćirić operator, fuzzy metric space, fixed point, nonlinear differential equations.

AMS Subject Classification: 54H25; 47H10; 45D05.

1 Introduction and preliminaries

The contraction mapping principle appeared in the explicit form in Banach's thesis in 1922 [1], where it was used to establish the existence of solution of integral equations. It states that if (X, d) is a complete metric space and $f\colon X \to X$ satisfies: there exists $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq \lambda d(x, y),\ \forall\, x, y \in X, \tag{1.1}$$

then f has a unique fixed point in X and, for each $x_0 \in X$, the sequence of iterates $\{f^n x_0\}$ converges to this fixed point. Since then, because of its simplicity and usefulness, this principle has become a very popular tool in solving existence problems in many branches of mathematical analysis. Prešić [16, 17] extended the Banach's principle to product spaces and used this extension to establish the convergence of some particular sequences.

Let k be a positive integer and $T\colon X^k \to X$ be a mapping. Then a point $x \in X$ is called a fixed point of T if $T(x, x, \dots, x) = x$. Prešić in his papers [16, 17] extended the Banach's contractive condition, that is, the contractivity condition (1.1) for the mapping $T\colon X^k \to X$, namely, he used the following condition: there exist nonnegative constants $\alpha_i, 1 \le i \le k$, such that $\sum_{i=1}^k \alpha_i < 1$, $d(T(x_1, \dots, x_k), T(x_2, \dots, x_{k+1})) \le \sum_{i=1}^k \alpha_i d(x_i, x_{i+1})$, for all $x_1, x_2, \dots, x_k, x_{k+1} \in X$, and proved a fixed point result for the mappings satisfying this condition.

The Prešić's theorem and its generalizations have various applications (see, for instance, [2, 6, 11, 16, 17, 19, 20] and the references therein). Ćirić [3] generalized Prešić's condition by considering the following condition: there exists $\lambda \in [0, 1)$ such that $d(T(x_1, \dots, x_k), T(x_2, \dots, x_{k+1})) \le \lambda \max_{1 \le i \le k} d(x_i, x_{i+1})$, for all $x_1, x_2, \dots, x_k, x_{k+1} \in X$.

On the other hand, fuzzy sets were first introduced by Zadeh [23] in 1965. He studied the uncertainties occurring in the behaviour of systems of stochastic nature by means of fuzzy sets. Kramosil and Michálek [12] used the concept of fuzzy set to define the metric in form of fuzzy sets and introduced the notion of fuzzy metric spaces. The fixed point theory on fuzzy metric spaces was introduced by Grabiec in [7], where a fuzzy metric version of Banach contraction principle was proved. However, it is important to note that no method is available to obtain metric Banach contraction from Grabiec fuzzy contraction. In 2002, Gregori and Sapena [9] introduced the notion of fuzzy contractive mapping and established Banach contraction theorem in various classes of complete fuzzy metric spaces in the sense of George and Veeramani [4], Kramosil and Michálek [12] and Grabiec [7]. The results obtained by Gregori and Sapena [9] have become recently of interest to many authors (see [5, 8, 13, 14, 15, 21, 22]).

Following this direction of research, we extend and generalize here the fuzzy contractive mappings of Gregori and Sapena [9] by introducing fuzzy-Prešić-Ćirić operators and prove some fixed point results for such operators in fuzzy metric spaces under H-type t-norms. The main theorem in this paper is illustrated with an example. Moreover, some applications to nonlinear differential equations are given to show the usability of the obtained results.

First, we recall some definitions and results which will be needed in the sequel.

Definition 1 [Schweizer and Sklar [18]]. A mapping $*\colon [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangular norm (t-norm for short) if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative, that is, $a * b = b * a$ and $a * (b * c) = (a * b) * c$, for all $a, b, c \in [0, 1]$;

(ii) $*$ is continuous;

(iii) $1 * a = a$, for all $a \in [0,1]$;

(iv) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

Some basic examples of t-norm are the minimum t-norm $*_m$, $a *_m b = \min\{a, b\}$, product t-norm $*_p$, $a *_p b = ab$, the Łukasiewicz t-norm $*_L$, $a *_L b = \max\{a + b - 1, 0\}$, for all $a, b \in [0,1]$.

Let $*$ be a given t-norm. For $a_1, a_2, \dots, a_n \in [0,1]$, we use the notation $*_{i=1}^{n} a_i = a_1 * a_2 * \cdots * a_n$.

Let $a \in [0,1]$. Then we can define the sequence $\{*^n a\}_{n\in\mathbb{N}}$ by $*^1 a = a$ and $*^{n+1} a = (*^n a) * a$, for $n \geq 1$.

Definition 2 [Hadžić and Pap [10]]. A t-norm T is said to be of H-type if the sequence $\{*^n a\}_{n\in\mathbb{N}}$ is equicontinuous at 1, that is, for all $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that, $a \in (1-\delta, 1]$ implies $*^n a > 1 - \varepsilon$ for all $n \in \mathbb{N}$.

An important H-type t-norm is $*_m$. Some other examples of H-type t-norms can be found in [10]. We denote the class of all Hadžić-type t-norms by $\mathcal{H}$.

Definition 3 [George and Veeramani [4]]. A triple $(X, M, *)$ is called a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and $M\colon X^2 \times (0,\infty) \to [0,1]$ is a fuzzy set satisfying the following conditions:

(GV1) $M(x, y, t) > 0$, for all $x, y \in X$ and $t > 0$;

(GV2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(GV3) $M(x, y, t) = M(y, x, t)$, for all $x, y \in X$ and $t > 0$;

(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$, for all $x, y, z \in X$ and $s, t > 0$;

(GV5) $M(x, y, \cdot)\colon (0,\infty) \to [0,1]$ is a continuous mapping, for all $x, y \in X$.

For the topological properties of a fuzzy metric space, the reader is referred to [4].

Definition 4 [George and Veeramani [4]; Schweizer and Sklar [18]]. Let $(X, M, *)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called an M-Cauchy sequence if for all $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all $n, m > n_0$. On the other hand, $\{x_n\}$ is called a G-Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+m}, t) = 1$ for each $m \in \mathbb{N}$ and $t > 0$ or, equivalently, $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

The sequence $\{x_n\}$ is called convergent and converges to x if, for all $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$, for all $n > n_0$.

We say that the space $(X, M, *)$ is M-complete (resp., G-complete) if every M-Cauchy (resp., G-Cauchy) sequence in X is convergent to some $x \in X$.

Theorem 1 [George and Veeramani [4]]. *Let $(X, M, *)$ be a fuzzy metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ converges to x if and only if*

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \forall\, t > 0.$$

Remark 1 [George and Veeramani [4]]. Let (X,d) be a metric space, then $(X,M_d,*_p)$ is a fuzzy metric space, where $M_d(x,y,t)=\dfrac{t}{t+d(x,y)}$, for all $x,y\in X$ and $t>0$. We call this M_d the standard fuzzy metric induced by the metric d. Further, (X,d) is complete if and only if $(X,M_d,*_p)$ is M-complete.

Note that the above result remains true if we take $*_m$ instead of $*_p$. We call the space $(X,M_d,*_m)$ the min-fuzzy metric space induced by d.
In the next three sections, we state our main results.

2 Fixed point theorems

First, we define Prešić-Ćirić operators in the framework of fuzzy metric spaces.

Definition 5. Let $(X,M,*)$ be a fuzzy metric space, k a positive integer and $T\colon X^k\to X$ be a mapping. Then T is called a fuzzy-Prešić-Ćirić operator if

$$\frac{1}{M(T(x_1,\dots,x_k),T(x_2,\dots,x_{k+1}),t)}-1\le\lambda\max_{1\le i\le k}\left\{\frac{1}{M(x_i,x_{i+1},t)}-1\right\},\quad (2.1)$$

for all $x_1,\dots,x_k,x_{k+1}\in X$ and $t>0$, where $\lambda\in(0,1)$. Alternatively, the above condition may be written as

$$M(T(x_1,\dots,x_k),T(x_2,\dots,x_{k+1}),t)\ge\left[\lambda\cdot\max_{1\le i\le k}\left\{\frac{1}{M(x_i,x_{i+1},t)}-1\right\}+1\right]^{-1},\quad (2.2)$$

for all $x_1,\dots,x_k,x_{k+1}\in X$ and $t>0$, where $\lambda\in(0,1)$.

Remark 2. Taking M as M_d in condition (2.1) (or (2.2)), we get

$$d(T(x_1,\dots,x_k),T(x_2,\dots,x_{k+1}))\le\lambda\max\{d(x_i,x_{i+1})\colon 1\le i\le k\},$$

for all $x_1,\dots,x_k,x_{k+1}\in X$, where $\lambda\in(0,1)$.

Next, we prove a fixed point theorem for the fuzzy-Prešić-Ćirić operators in M-complete fuzzy metric spaces.

Theorem 2. *Let $(X,M,*)$ be an M-complete fuzzy metric space, k a positive integer and $T\colon X^k\to X$ a fuzzy-Prešić-Ćirić operator. Suppose that one of the following conditions holds:*

(H1) $*\in\mathcal{H}$ *and there exist $x_1,x_2\dots,x_k\in X$ such that*

$$\inf_{t>0}M(x_i,x_{i+1},t)>0,\ i=1,2,\dots,k-1,\ \inf_{t>0}M(x_k,T(x_1,\dots,x_k),t)>0.$$

(H2) *There exist $x_1,x_2\dots,x_k\in X$ such that the following property holds: for each $\varepsilon\in(0,1)$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that, for $m,n>n_0$, with $m>n$, we get $*_{j=n}^{m-1}\left[1+\mu(ts_j^{(n)})\lambda^{\frac{j}{k}}\right]^{-1}>1-\varepsilon$, where $s_j^{(n)}=\frac{1}{2^{j-n+1}}$, $j=n,\dots,m-2$, $s_{m-1}^{(n)}=s_{m-2}^{(n)}$ and $\mu(z):=\max\left\{\max_{1\le i\le k-1}\frac{1}{\lambda^{\frac{i}{k}}}\times\left[\frac{1}{M(x_i,x_{i+1},z)}-1\right],\frac{1}{\lambda}\left[\frac{1}{M(x_k,T(x_1,\dots,x_k),z)}-1\right]\right\}$.*

Then T has a fixed point in X. If, in addition, we suppose that, on the diagonal $\Delta \subset X^k$,

$$M(T(u,\ldots,u),T(v,\ldots,v),t) > M(u,v,t), \quad \forall\, t>0 \tag{2.3}$$

holds for $u,v \in X$ with $u \neq v$, then T has a unique fixed point.

Proof. Let $x_1, x_2, \ldots, x_k$ be the points in X given by hypothesis. Define a sequence $\{x_n\}$ by $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, $\forall\, n \in \mathbb{N}$. For notational convenience, set $M(x_n, x_{n+1}, t) = M_n(t)$, for all $n \in \mathbb{N}$ and $t > 0$ and consider $\mu(t) := \max\left\{\frac{1}{\theta^i}\left[\frac{1}{M_i(t)} - 1\right] : 1 \le i \le k\right\}$, where $\theta = \lambda^{\frac{1}{k}}$. By mathematical induction, we show that

$$1/M_n(t) - 1 \le \mu(t)\theta^n, \quad \forall\, n \in \mathbb{N},\ \forall\, t>0. \tag{2.4}$$

By the definition of $\mu(t)$, it is obvious that (2.4) is true for $n = 1, 2, \ldots, k$. Let the following k inequalities, for $t > 0$, $\frac{1}{M_n(t)} - 1 \le \mu(t)\theta^n$, $\frac{1}{M_{n+1}(t)} - 1 \le \mu(t)\theta^{n+1}, \ldots, \frac{1}{M_{n+k-1}(t)} - 1 \le \mu(t)\theta^{n+k-1}$ be the induction hypothesis. Since $\theta = \lambda^{\frac{1}{k}} < 1$, from (2.1) we have

$$\begin{aligned}
\frac{1}{M_{n+k}(t)} - 1 &= \frac{1}{M(T(x_n, x_{n+1}, \ldots, x_{n+k-1}), T(x_{n+1}, x_{n+2}, \ldots, x_{n+k}), t)} - 1\\
&\le \lambda \max\left\{\frac{1}{M(x_{n+i-1}, x_{n+i}, t)} - 1 \colon 1 \le i \le k\right\}\\
&= \lambda \max\left\{\frac{1}{M_{n+i-1}(t)} - 1 \colon 1 \le i \le k\right\}\\
&\le \lambda \max\left\{\mu(t)\,\theta^{n+i-1} \colon 1 \le i \le k\right\} \le \lambda\mu(t)\theta^n = \mu(t)\theta^{n+k},\ t>0.
\end{aligned}$$

Hence, by induction, (2.4) is true for all $n \in \mathbb{N}$.

Next, we show that $\{x_n\}$ is an M-Cauchy sequence. Consider $\varepsilon \in (0,1)$ and $t > 0$ fixed. For $n, m \in \mathbb{N}$ with $m > n$, we have, using (2.4), that

$$\begin{aligned}
M(x_n, x_m, t) \ge& M(x_n, x_{n+1}, t/2) * M(x_{n+1}, x_m, t/2)\\
\ge& M(x_n, x_{n+1}, t/2) * M(x_{n+1}, x_{n+2}, t/2^2)\\
&* \cdots * M(x_{m-2}, x_{m-1}, t/2^{m-n-1}) * M(x_{m-1}, x_m, t/2^{m-n-1})\\
=& \left(*_{j=n}^{m-2} M_j(t/2^{j-(n-1)})\right) * M_{m-1}(t/2^{m-n-1})\\
\ge& \left(*_{j=n}^{m-2}\left[1 + \mu(t/2^{j-(n-1)})\theta^j\right]^{-1}\right) * \left[1 + \mu(t/2^{m-n-1})\theta^{m-1}\right]^{-1}\\
\ge& \left(*_{j=n}^{m-2}\left[1 + \mu(t/2^{j-(n-1)})\theta^n\right]^{-1}\right) * \left[1 + \mu(t/2^{m-n-1})\theta^{n}\right]^{-1}.
\end{aligned}$$

Under condition (H1), it is easy to check that $\mu := \sup_{t>0} \mu(t) \in [0, \infty)$, since

$$\sup_{t>0}\mu(t) \le \max_{1\le i\le k}\left\{\frac{1}{\theta^i}\sup_{t>0}\left[\frac{1}{M_i(t)} - 1\right]\right\} = \max_{1\le i\le k}\left\{\frac{1}{\theta^i}\left[\frac{1}{\inf_{t>0} M_i(t)} - 1\right]\right\}$$

and $\inf_{t>0} M_i(t) = \inf_{t>0} M(x_i, x_{i+1}, t) > 0$, $i = 1, 2, \ldots, k-1$,

$$\inf_{t>0} M_k(t) = \inf_{t>0} M(x_k, x_{k+1}, t) = \inf_{t>0} M(x_k, T(x_1, \ldots, x_k), t) > 0.$$

In these conditions, for $n, m \in \mathbb{N}$ with $m > n$, we have that

$$M(x_n, x_m, t) \geq [1 + \mu\theta^n]^{-1} * \cdots * [1 + \mu\theta^n]^{-1} = *^{m-n} [1 + \mu\theta^n]^{-1}. \tag{2.5}$$

Since $* \in \mathcal{H}$, there exists $\delta \in (0,1)$ such that: if $[1 + \mu\theta^n]^{-1} \in (1-\delta, 1]$, then $*^{m-n} [1 + \mu\theta^n]^{-1} > 1 - \varepsilon$, for $m > n$. As $0 < \theta < 1$, given $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $[1 + \mu\theta^n]^{-1} \in (1-\delta, 1]$, for all $n > n_0$ (it suffices to take $n_0 \in \mathbb{N}$ such that $1 + \mu\theta^{n_0} < \frac{1}{1-\delta}$). With this choice, we obtain $*^{m-n} [1 + \mu\theta^n]^{-1} > 1 - \varepsilon$, $\forall\, n > n_0$, $m > n$. The above inequality with (2.5) and the properties of M give $M(x_n, x_m, t) > 1 - \varepsilon$, $\forall\, n, m > n_0$, $\forall\, t > 0$.

On the other hand, if (H2) holds, there exists $n_0 \in \mathbb{N}$ such that, for $m, n > n_0$, with $m > n$, we get

$$\begin{aligned}
M(x_n, x_m, t) &\geq \left(*_{j=n}^{m-2} \left[1 + \mu(t/2^{j-(n-1)})\theta^j\right]^{-1} \right) * \left[1 + \mu(t/2^{m-n-1})\theta^{m-1}\right]^{-1} \\
&= \mathop{*}_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n)})\theta^j\right]^{-1} > 1 - \varepsilon.
\end{aligned}$$

Thus, in both situations, $\{x_n\}$ is an M-Cauchy sequence. By M-completeness of X, there exists $u \in X$ such that

$$\lim_{n\to\infty} M(x_n, u, t) = 1, \ \forall\, t > 0. \tag{2.6}$$

Now, we show that u is a fixed point of T. Indeed, for any $n \in \mathbb{N}$ and $t > 0$, we have

$$\begin{aligned}
&M(x_{n+k}, T(u, \ldots, u), t) = M(T(x_n, \ldots, x_{n+k-1}), T(u, \ldots, u), t) \\
&\quad \geq M(T(x_n, \ldots, x_{n+k-1}), T(x_{n+1}, \ldots, x_{n+k-1}, u), t/2) \\
&\qquad * M(T(x_{n+1}, \ldots, x_{n+k-1}, u), T(x_{n+2}, \ldots, x_{n+k-1}, u, u), t/2^2) \\
&\qquad * \cdots * M(T(x_{n+k-2}, x_{n+k-1}, u, \ldots, u), T(x_{n+k-1}, u, \ldots, u), t/2^{k-1}) \\
&\qquad * M(T(x_{n+k-1}, u, \ldots, u), T(u, \ldots, u), t/2^{k-1}).
\end{aligned} \tag{2.7}$$

Using (2.2), (2.4) and $\theta \in (0,1)$, we have

$$\begin{aligned}
&M(T(x_n, \ldots, x_{n+k-1}), T(x_{n+1}, \ldots, x_{n+k-1}, u), t) \\
&\geq \left[\lambda \max\left\{ \max_{1\leq i\leq k-1} \left\{\frac{1}{M(x_{n+i-1}, x_{n+i}, t)} - 1\right\}, \frac{1}{M(x_{n+k-1}, u, t)} - 1\right\} + 1\right]^{-1} \\
&= \left[\lambda \max\left\{ \max_{1\leq i\leq k-1} \left\{\frac{1}{M_{n+i-1}(t)} - 1\right\}, \frac{1}{M(x_{n+k-1}, u, t)} - 1\right\} + 1\right]^{-1} \\
&\geq \left[\lambda \max\left\{ \max_{1\leq i\leq k-1} \left\{\mu(t)\theta^{n+i-1}\right\}, \frac{1}{M(x_{n+k-1}, u, t)} - 1\right\} + 1\right]^{-1} \\
&\geq \left[\lambda \max\left\{\mu(t)\theta^n, \frac{1}{M(x_{n+k-1}, u, t)} - 1\right\} + 1\right]^{-1}.
\end{aligned}$$

Using (2.6) and the fact that $0 < \theta < 1$, it follows from the above inequality that $\lim_{n\to\infty} M(T(x_n, \dots, x_{n+k-1}), T(x_{n+1}, \dots, x_{n+k-1}, u), t) = 1,\ \forall\ t > 0$. Similarly, $\lim_{n\to\infty} M(T(x_{n+1}, \dots, x_{n+k-1}, u), T(x_{n+2}, \dots, x_{n+k-1}, u, u), t) = 1,\ \forall\ t > 0, \dots,$ $\lim_{n\to\infty} M(T(x_{n+k-1}, u, \dots, u), T(u, \dots, u), t) = 1,\ \forall\ t > 0$. The above properties with (2.7) imply that

$$\lim_{n\to\infty} M(x_{n+k}, T(u, \dots, u), t) = 1,\ \forall\ t > 0. \tag{2.8}$$

Therefore, for any $n \in \mathbb{N}$ and $t > 0$, we have

$$M(u, T(u, u, \dots, u), t) \quad \geq \quad M(u, x_{n+k}, t/2) * M(x_{n+k}, T(u, \dots, u), t/2),$$

which, together with (2.6) and (2.8), gives $M(u, T(u, u, \dots, u), t) = 1, \forall\ t > 0$. Thus, $T(u, u, \dots, u) = u$, that is, u is a fixed point of T.

Finally, for uniqueness, suppose that $v \in X$ is another fixed point of T with $u \neq v$. Then, from (2.3), we have $M(u, v, t) = M(T(u, \dots, u), T(v, \dots, v), t) > M(u, v, t)$, for every $t > 0$. This contradiction shows that $u = v$. Thus, under condition (2.3), the fixed point of T is unique. □

Remark 3. The uniqueness condition (2.3) in Theorem 2 is reduced, for $M = M_d$, to $d(T(u, \dots, u), T(v, \dots, v)) < d(u, v)$, $\forall\ u, v \in X$ with $u \neq v$. From the proof of Theorem 2, it is obvious that uniqueness is also derived just considering the following weaker hypothesis:

$$\begin{aligned} &\text{for each } u, v \in X \text{ fixed with } u \neq v, \text{ there exists } t > 0 \text{ such that} \\ &\qquad\qquad M(T(u, \dots, u), T(v, \dots, v), t) > M(u, v, t). \end{aligned} \tag{2.9}$$

Corollary 1. Let $(X, M, *)$ be an M-complete fuzzy metric space and $T\colon X \to X$ be a fuzzy contractive mapping (see Gregori and Sapena [9]), that is,

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq \lambda \left(\frac{1}{M(x, y, t)} - 1 \right),\ \forall x, y \in X,\ \forall t > 0,$$

where $\lambda \in (0, 1)$. Suppose that one of the following conditions holds:

(h1) $* \in \mathcal{H}$ and there exists $x_1 \in X$ such that $\inf_{t>0} M(x_1, T(x_1), t) > 0$.

(h2) There exists $x_1 \in X$ such that the following property holds: for each $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for $m, n > n_0$, with $m > n$, we get $*_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n)})\lambda^j\right]^{-1} > 1 - \varepsilon$, where $s_j^{(n)} = \frac{1}{2^{j-n+1}}$, $j = n, \dots, m-2$, $s_{m-1}^{(n)} = s_{m-2}^{(n)}$ and $\mu(z) := \frac{1}{\lambda} \left[\frac{1}{M(x_1, T(x_1), z)} - 1\right]$.

Then T has a unique fixed point in X.

Proof. Take $k = 1$ in Theorem 2, then the existence of a fixed point $u \in X$ follows. Further, for $x,\ y \in X$ fixed with $x \neq y$, we get $\frac{1}{M(x,y,t)} - 1 > 0$ for some $t > 0$, thus, since T is a fuzzy contractive mapping, we have $\frac{1}{M(Tx,Ty,t)} - 1 < \frac{1}{M(x,y,t)} - 1$, for some $t > 0$, that is, $M(Tx, Ty, t) > M(x, y, t)$, for some $t > 0$, where $x, y \in X$ are fixed with $x \neq y$. Hence, the uniqueness condition (2.9) of Remark 3 is satisfied. Therefore, the fixed point of T is unique. □

Remark 4. Note that, if we select $M = M_d$, conditions (H1) in Theorem 2 and (h1) in Corollary 1 are valid only if the mapping T has fixed points. Indeed, for $k = 1$, $\inf_{t>0} M_d(x_1, T(x_1), t) = \inf_{t>0} \frac{t}{t+d(x_1,T(x_1))} = 1$, if $T(x_1) = x_1$ and $\inf\limits_{t>0} M_d(x_1, T(x_1), t) = 0$, if $T(x_1) \neq x_1$. Similarly, in the general case $k \in \mathbb{N}$, for the validity of (H1) it is obliged that $x_1 = x_2 = \cdots = x_k = T(x_1, x_2, \ldots, x_k)$. Thus, for the standard fuzzy metric induced by the metric d, the restriction (H1) is not useful, since it requires to start the process with a fixed point.

However, condition (H2) is interesting in the general case and, in particular, for the standard fuzzy metric induced by the metric d. Indeed, for $k = 1$, $M = M_d$ and any $x_1 \in X$, we have

$$\mu(z) := \frac{1}{\lambda}\Big[\frac{z + d(x_1, T(x_1))}{z} - 1\Big] = \frac{d(x_1, T(x_1))}{\lambda z}$$

and, thus, for $t > 0$ and $m, n \in \mathbb{N}$ with $m > n$, $*_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n)})\lambda^j\right]^{-1} = *_{j=n}^{m-1} \left[1 + \frac{d(x_1,T(x_1))}{t} \frac{\lambda^{j-1}}{s_j^{(n)}}\right]^{-1}$. Note that, if $T(x_1) = x_1$, then μ is null and, for every $\varepsilon > 0$, it is satisfied that $\mathop{*}\limits_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n)})\lambda^j\right]^{-1} = 1 > 1 - \varepsilon$, for all $m, n \in \mathbb{N}$ with $m > n$. Consider the general case $T(x_1) \neq x_1$ or $T(x_1) = x_1$. If we take $* = *_m$, then, for $t > 0$ and $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}
\mathop{*}\limits_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n)})\lambda^j\right]^{-1} &= \min_{n \le j \le m-1} \left[1 + \frac{d(x_1, T(x_1))}{t} \frac{\lambda^{j-1}}{s_j^{(n)}}\right]^{-1} \\
&= \left[1 + \frac{d(x_1, T(x_1))}{t} \max_{n \le j \le m-1} \frac{\lambda^{j-1}}{s_j^{(n)}}\right]^{-1}.
\end{aligned}$$

Replacing $s_j^{(n)} = \frac{1}{2^{j-n+1}}$, $j = n, \ldots, m-2$ and $s_{m-1}^{(n)} = s_{m-2}^{(n)}$, we have for $t > 0$ and $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}
&\max_{n \le j \le m-1} \frac{\lambda^{j-1}}{s_j^{(n)}} = \max\left\{2\lambda^{n-1}, 4\lambda^n, \ldots, 2^{m-n-1}\lambda^{m-3}, 2^{m-n-1}\lambda^{m-2}\right\} \\
&= \max\left\{(2\lambda)^{n-1}2^{-n+2}, (2\lambda)^n 2^{-n+2}, \ldots, (2\lambda)^{m-3}2^{-n+2}, (2\lambda)^{m-2}2^{-n+1}\right\} \\
&\le \max\left\{(2\lambda)^{n-1}2^{-n+2}, (2\lambda)^n 2^{-n+2}, \ldots, (2\lambda)^{m-3}2^{-n+2}, (2\lambda)^{m-2}2^{-n+2}\right\} \\
&\le \max\left\{(2\lambda)^{n-1}, (2\lambda)^n, \ldots, (2\lambda)^{m-3}, (2\lambda)^{m-2}\right\} 2^{-n+2}.
\end{aligned}$$

If $\lambda \in (0, \frac{1}{2}]$, since $m > n$, we get $0 \le \max\limits_{n \le j \le m-1} \lambda^{j-1}/s_j^{(n)} \le (2\lambda)^{n-1}2^{-n+2} = 2\lambda^{n-1} \to 0$, as $n \to \infty$, so that, for $\varepsilon \in (0, 1)$ and $t > 0$ fixed, there exists $n_0 \in \mathbb{N}$ such that, for $m, n > n_0$, with $m > n$, $\max\limits_{n \le j \le m-1} \lambda^{j-1}/s_j^{(n)} < \dfrac{t}{d(x_1, T(x_1))} \dfrac{\varepsilon}{1 - \varepsilon}$ and, hence,

$$\mathop{*}\limits_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n)})\lambda^j\right]^{-1} = \left[1 + \frac{d(x_1, T(x_1))}{t} \max_{n \le j \le m-1} \lambda^{j-1}/s_j^{(n)}\right]^{-1} > 1 - \varepsilon.$$

This proves that condition (h2) holds for $M = M_d$ and $* = *_m$ if $\lambda \in (0, \frac{1}{2}]$, independently of the choice of $x_1 \in X$. Moreover, for a general $k \in \mathbb{N}$, taking $M = M_d$ and any $x_1, \dots, x_k \in X$, we get

$$\begin{aligned}
\mu(z) &:= \max\Big\{ \max_{1\le i\le k-1} \frac{1}{\lambda^{\frac{i}{k}}}\Big[\frac{z+d(x_i,x_{i+1})}{z}-1\Big], \frac{1}{\lambda}\Big[\frac{z+d(x_k,T(x_1,\dots,x_k))}{z}-1\Big]\Big\} \\
&= \max\Big\{ \max_{1\le i\le k-1} \frac{1}{\lambda^{\frac{i}{k}} z} d(x_i,x_{i+1}), \frac{1}{\lambda z} d(x_k,T(x_1,\dots,x_k))\Big\} \\
&= \frac{1}{z}\max\Big\{ \max_{1\le i\le k-1} \frac{d(x_i,x_{i+1})}{\lambda^{\frac{i}{k}}}, \frac{d(x_k,T(x_1,\dots,x_k))}{\lambda}\Big\}.
\end{aligned}$$

Hence, for $t > 0$ and $m, n \in \mathbb{N}$ with $m > n$, $*_{j=n}^{m-1}\Big[1+\mu(ts_j^{(n)})\lambda^{\frac{j}{k}}\Big]^{-1}$ is equal to

$$\mathop{*}_{j=n}^{m-1}\left[1+\frac{1}{ts_j^{(n)}}\max\left\{\max_{1\le i\le k-1}\frac{d(x_i,x_{i+1})}{\lambda^{\frac{i}{k}}}, \frac{d(x_k,T(x_1,\dots,x_k))}{\lambda}\right\}\lambda^{\frac{j}{k}}\right]^{-1}.$$

Taking $* = *_m$, we have, for $t > 0$ and $m, n \in \mathbb{N}$ with $m > n$, that the previous expression is

$$\left[1+\frac{1}{t}\max_{n\le j\le m-1}\frac{1}{s_j^{(n)}}\max\left\{\max_{1\le i\le k-1}\frac{d(x_i,x_{i+1})}{\lambda^{\frac{i}{k}}}, \frac{d(x_k,T(x_1,\dots,x_k))}{\lambda}\right\}\lambda^{\frac{j}{k}}\right]^{-1}.$$

Now, since $\lambda > 0$ and $s_j^{(n)} > 0$, for every $j = n, \dots, m-1$, we get

$$\begin{aligned}
&\max_{n\le j\le m-1}\frac{1}{s_j^{(n)}}\max\left\{\max_{1\le i\le k-1}\frac{d(x_i,x_{i+1})}{\lambda^{\frac{i}{k}}}, \frac{d(x_k,T(x_1,\dots,x_k))}{\lambda}\right\}\lambda^{\frac{j}{k}} \\
&= \max_{n\le j\le m-1}\max\left\{\max_{1\le i\le k-1}\frac{d(x_i,x_{i+1})\lambda^{\frac{j}{k}}}{s_j^{(n)}\lambda^{\frac{i}{k}}}, \frac{d(x_k,T(x_1,\dots,x_k))\lambda^{\frac{j}{k}}}{s_j^{(n)}\lambda}\right\} \\
&\le \max\left\{\max_{1\le i\le k-1}\max_{n\le j\le m-1}\frac{d(x_i,x_{i+1})\lambda^{\frac{j}{k}}}{s_j^{(n)}\lambda^{\frac{i}{k}}}, \max_{n\le j\le m-1}\frac{d(x_k,T(x_1,\dots,x_k))\lambda^{\frac{j}{k}}}{s_j^{(n)}\lambda}\right\} \\
&= \max\left\{\max_{1\le i\le k-1}\frac{d(x_i,x_{i+1})}{\lambda^{\frac{i}{k}}}\max_{n\le j\le m-1}\frac{\lambda^{\frac{j}{k}}}{s_j^{(n)}}, \frac{d(x_k,T(x_1,\dots,x_k))}{\lambda}\max_{n\le j\le m-1}\frac{\lambda^{\frac{j}{k}}}{s_j^{(n)}}\right\} \\
&= \max\left\{\max_{1\le i\le k-1}\frac{d(x_i,x_{i+1})}{\lambda^{\frac{i}{k}}}, \frac{d(x_k,T(x_1,\dots,x_k))}{\lambda}\right\}\max_{n\le j\le m-1}\frac{\lambda^{\frac{j}{k}}}{s_j^{(n)}}.
\end{aligned}$$

For $s_j^{(n)} = \frac{1}{2^{j-n+1}}$, $j = n, \dots, m-2$ and $s_{m-1}^{(n)} = s_{m-2}^{(n)}$, we have for $t > 0$ and $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}
&\max_{n\le j\le m-1}\frac{\lambda^{\frac{j}{k}}}{s_j^{(n)}} = \max\left\{2\lambda^{\frac{n}{k}}, 4\lambda^{\frac{n+1}{k}}, \dots, 2^{m-n-1}\lambda^{\frac{m-2}{k}}, 2^{m-n-1}\lambda^{\frac{m-1}{k}}\right\} \\
&= \max\left\{(2\lambda^{\frac{1}{k}})^n 2^{-n+1}, (2\lambda^{\frac{1}{k}})^{n+1}2^{-n+1}, \dots, (2\lambda^{\frac{1}{k}})^{m-2}2^{-n+1}, (2\lambda^{\frac{1}{k}})^{m-1}2^{-n}\right\} \\
&\le \max\left\{(2\lambda^{\frac{1}{k}})^n, (2\lambda^{\frac{1}{k}})^{n+1}, \dots, (2\lambda^{\frac{1}{k}})^{m-2}, (2\lambda^{\frac{1}{k}})^{m-1}\right\}2^{-n+1}.
\end{aligned}$$

If $\lambda \in (0, \frac{1}{2^k}]$, then $2\lambda^{\frac{1}{k}} \leq 1$ and, since $m > n$, we get $0 \leq \max_{n \leq j \leq m-1} \frac{\lambda^{\frac{j}{k}}}{s_j^{(n)}} \leq (2\lambda^{\frac{1}{k}})^n 2^{-n+1} = 2\lambda^{\frac{n}{k}} \to 0$, as $n \to \infty$, so that, for $\varepsilon \in (0,1)$ and $t > 0$ fixed, there exists $n_0 \in \mathbb{N}$ such that, for m, $n > n_0$, with $m > n$,

$$\max_{n \leq j \leq m-1} \frac{1}{s_j^{(n)}} \max\left\{ \max_{1 \leq i \leq k-1} \frac{d(x_i, x_{i+1})}{\lambda^{\frac{i}{k}}}, \frac{d(x_k, T(x_1, \ldots, x_k))}{\lambda} \right\} \lambda^{\frac{j}{k}} < t\frac{\varepsilon}{1-\varepsilon}$$

and $*_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n)})\lambda^{\frac{j}{k}}\right]^{-1} > 1 - \varepsilon$. Therefore, (H2) holds for $M = M_d$ and $* = *_m$ if $\lambda \in (0, \frac{1}{2^k}]$, independently of the choice of $x_1, \ldots, x_k \in X$.

Next, we give an example which illustrates Theorem 2.

Example 1. Let $0 < \alpha < \frac{1}{2}$, $x_n = \frac{2\alpha + 2^{n-1} - 1}{2^n}, n \in \mathbb{N}$, and consider the set $X = \{1\} \cup \{x_n : n \in \mathbb{N}\}$. Define the fuzzy set $M \colon X^2 \times (0, \infty) \to [0,1]$ by

$$M(x,y,t) = \begin{cases} x *_m y, & \text{if } x \neq y; \\ 1, & \text{if } x = y; \end{cases} \quad \forall x, y \in X,\ \forall t > 0.$$

Then $(X, M, *_m)$ is an M-complete fuzzy metric space and $*_m \in \mathcal{H}$.

For $k = 2$, let $T \colon X^2 \to X$ be defined by

$$T(x,y) = \begin{cases} 2(x_i *_m x_j), & \text{if } x = x_i,\ y = x_j,\ i < j; \\ 1, & \text{otherwise.} \end{cases}$$

Now, by some routine calculations, one can see that T satisfies

$$\frac{1}{M(T(y_1,y_2), T(y_2,y_3), t)} - 1 \leq \lambda \max\left\{ \frac{1}{M(y_1,y_2,t)} - 1, \frac{1}{M(y_2,y_3,t)} - 1 \right\},$$

for all $y_1, y_2, y_3 \in X$ and $t > 0$, where $\lambda = \frac{1}{2}$. Therefore, T is a fuzzy-Prešić-Ćirić operator with $\lambda = 1/2$.

We prove that condition (H1) is satisfied. Indeed, $*_m \in \mathcal{H}$ and, besides, starting with the points $x_1 = \alpha < x_2 = \frac{1}{4} + \frac{\alpha}{2}$ in X, we get $\inf_{t>0} M(x_1, x_2, t) = \min\{x_1, x_2\} = x_1 = \alpha > 0$ and

$$\begin{aligned} \inf_{t>0} M(x_2, T(x_1,x_2), t) &= \inf_{t>0} M(x_2, 2\min\{x_1,x_2\}, t) = \inf_{t>0} M\left(\frac{1}{4} + \frac{\alpha}{2}, 2\alpha, t\right) \\ &= \begin{cases} 1, & \text{if } \alpha = \frac{1}{6}, \\ \min\{\frac{1}{4} + \frac{\alpha}{2}, 2\alpha\}, & \text{if } \alpha \neq \frac{1}{6} \end{cases} > 0. \end{aligned}$$

Also, by definition of T, for $x, y \in X$ with $x \neq y$, we have

$$M(T(x,x), T(y,y), t) = M(1,1,t) = 1 > M(x,y,t), \quad \forall t > 0.$$

Hence, all the conditions of Theorem 2 are satisfied and, thus, we can conclude the existence of a unique fixed point of T. In fact, 1 is the unique fixed point of T.

Next, we give a sufficient condition for the validity of condition (2.3) under hypothesis (2.1) (or, equivalently, (2.2)) provided that $k \geq 2$. This condition is related to M and the t-norm selected $*$ and allows to establish the following corollary of Theorem 2.

Corollary 2. Let $(X, M, *)$ be an M-complete fuzzy metric space, k an integer with $k \geq 2$ and $T\colon X^k \to X$ a fuzzy-Prešić-Ćirić operator. Suppose that one of the conditions (H1) or (H2) holds. Then T has a fixed point in X. If, in addition, we suppose that

$$\text{for each } u,\ v \in X \text{ fixed with } u \neq v, \text{ there exists } t > 0 \text{ such that}$$
$$\mathop{*}_{i=1}^{k} \left[\lambda\left(1/z_i - 1\right) + 1\right]^{-1} > M(u, v, t), \tag{2.10}$$

where $z_i = M(u, v, t/2^i)$, for $i = 1, \dots, k-1$, and $z_k = z_{k-1}$, then T has a unique fixed point.

Proof. The first part of the corollary follows from the proof of Theorem 2. Now, for the uniqueness of fixed point, suppose that u, $v \in X$ are fixed points of T with $u \neq v$. Then, for any $t > 0$, we have

$$\begin{aligned} M(u,v,t) &= M(T(u,\dots,u), T(v,\dots,v), t) \qquad (2.11)\\ &\geq M(T(u,\dots,u), T(u,\dots,u,v), t/2) * M(T(u,\dots,u,v), T(v,\dots,v), t/2)\\ &\geq M(T(u,\dots,u), T(u,\dots,u,v), t/2) * M(T(u,\dots,u,v), T(u,\dots,u,v,v), t/2^2)\\ &\quad * \cdots * M(T(u,u,v,\dots,v), T(u,v,\dots,v), t/2^{k-1})\\ &\quad * M(T(u,v,\dots,v), T(v,\dots,v), t/2^{k-1}). \end{aligned}$$

Using (2.2), we get, for every $t > 0$, $M(T(u,\dots,u), T(u,\dots,u,v), t/2) \geq \left[\lambda\left(\frac{1}{M(u,v,t/2)} - 1\right) + 1\right]^{-1}$ and, similarly,
$M(T(u,\dots,u,v), T(u,\dots,u,v,v), t/2^2) \geq \left[\lambda\left(\frac{1}{M(u,v,t/2^2)} - 1\right) + 1\right]^{-1}, \dots,$
$M(T(u,u,v,\dots,v), T(u,v,\dots,v), t/2^{k-1}) \geq \left[\lambda\left(\frac{1}{M(u,v,t/2^{k-1})} - 1\right) + 1\right]^{-1}$
$M(T(u,v,\dots,v), T(v,\dots,v), t/2^{k-1}) \geq \left[\lambda\left(\frac{1}{M(u,v,t/2^{k-1})} - 1\right) + 1\right]^{-1}.$
In consequence, by (2.11), for every $t > 0$, the following inequality holds

$$\begin{aligned} M(u,v,t) \geq{}& \left[\lambda\left(\frac{1}{M(u,v,t/2)} - 1\right) + 1\right]^{-1} * \left[\lambda\left(\frac{1}{M(u,v,t/2^2)} - 1\right) + 1\right]^{-1}\\ &* \cdots * \left[\lambda\left(\frac{1}{M(u,v,t/2^{k-1})} - 1\right) + 1\right]^{-1} * \left[\lambda\left(\frac{1}{M(u,v,t/2^{k-1})} - 1\right) + 1\right]^{-1}. \end{aligned}$$

From the previous inequality and (2.10), we can affirm that there exists $t > 0$ such that $M(u,v,t) > M(u,v,t)$, which is a contradiction, so that $u = v$ and the fixed point of T is unique. $\square$

Next, we present the following extension of Theorem 2, which is very interesting to the applications included in the last section.

Theorem 3. *Let $(X, M, *)$ be an M-complete fuzzy metric space, k a positive integer and $T\colon X^k \to X$ a fuzzy-Prešić-Ćirić operator. Suppose that one of the following conditions holds:*

(H1*) *Condition (H1).*

(H2*) *There exist $x_1, x_2 \dots, x_k \in X$ such that the following property holds: for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for $m, n > n_0$, with $m > n$, we get $*_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n,m)})\lambda^{\frac{j}{k}}\right]^{-1} > 1 - \varepsilon$, for some collection of values $s_j^{(n,m)} > 0$, $j = n, \dots, m-1$, with $\sum_{j=n}^{m-1} s_j^{(n,m)} \le 1$, where $\mu(z)$ is given in the statement of Theorem 2.*

Then T has a fixed point in X. If, in addition, we suppose that, on the diagonal $\Delta \subset X^k$, condition (2.3) holds for $u, v \in X$ with $u \neq v$, then T has a unique fixed point.

Proof. It is similar to the proof of Theorem 2. In order to prove that $\{x_n\}$ is an M-Cauchy sequence, we consider the choice for $s_j^{(n,m)}$ given in the statement. Using the nondecreasing character of $M(x, y, \cdot)$ for every x, $y \in X$ and following the proof of Theorem 4.8 [9], we have, for $t > 0$ and $n, m \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} M(x_n, x_m, t) &\ge M(x_n, x_{n+1}, ts_n^{(n,m)}) * M(x_{n+1}, x_{n+2}, ts_{n+1}^{(n,m)}) \\ &* \cdots * M(x_{m-2}, x_{m-1}, ts_{m-2}^{(n,m)}) * M(x_{m-1}, x_m, ts_{m-1}^{(n,m)}) \\ &= \mathop{*}_{j=n}^{m-1} M_j(ts_j^{(n,m)}) \ge \mathop{*}_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n,m)})\theta^j\right]^{-1}. \end{aligned}$$

The case (H1*) is analogous to the proof of Theorem 2. Under condition (H2*), given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for m, $n > n_0$ with $m > n$, we get $M(x_n, x_m, t) \ge *_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n,m)})\theta^j\right]^{-1} > 1 - \varepsilon$. The proof is completed similarly to that of Theorem 2. □

Corollary 3. Let $(X, M, *)$ be an M-complete fuzzy metric space and $T\colon X \to X$ be a fuzzy contractive mapping, that is, $\frac{1}{M(Tx,Ty,t)} - 1 \le \lambda \left(\frac{1}{M(x,y,t)} - 1\right)$, $\forall x, y \in X$, $\forall t > 0$, where $\lambda \in (0,1)$. Suppose that one of the following conditions holds:

(h1*) Condition (h1) is satisfied.

(h2*) There exists $x_1 \in X$ such that the following property holds: for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for m, $n > n_0$, with $m > n$, we get $*_{j=n}^{m-1} \left[1 + \mu(ts_j^{(n,m)})\lambda^j\right]^{-1} > 1 - \varepsilon$, for some collection of values $s_j^{(n,m)} > 0$, $j = n, \dots, m-1$, with $\sum_{j=n}^{m-1} s_j^{(n,m)} \le 1$, where μ is given in Corollary 1.

Then T has a unique fixed point in X.

Remark 5. Theorem 3 and Corollary 3 are more general than Theorem 2 and Corollary 1, respectively. Conditions (H2*) and (h2*) show that the relevant point in the choice of the values $s_j^{(n,m)}$ is the fact that $s_j^{(n,m)} > 0$, for every $j = n, \dots, m-1$, and $\sum_{j=n}^{m-1} s_j^{(n,m)} \leq 1$. Hence, we can select different values of $s_j^{(n,m)}$ as long as these requirements are fulfilled. Note also that it is possible to take the expressions of $s_j^{(n,m)}$ to be independent of m, that is, in the form $s_j^{(n)}$, if we select them as positive numbers such that $\sum_{j=n}^{\infty} s_j^{(n)} \leq 1$. Moreover, we can take the expressions of $s_j^{(n,m)}$ to be independent of n and m, that is, in the form s_j, if we select them as positive numbers such that $\sum_{j=1}^{\infty} s_j \leq 1$.

Remark 6. It is important to note that, in the case where $M = M_d$ and $* = *_m$, condition (H2*) (resp. (h2*)) is trivially valid for arbitrary choices of $x_1, \dots, x_k$ (resp., x_1) and for any value of $\lambda \in (0,1)$, since we can choose $s_j^{(n,m)} = \frac{1}{j(j+1)}$, for $j = n, \dots, m-1$, which are positive and such that $\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = 1$.

We start with the case $k = 1$. For $M = M_d$, $* = *_m$, any $x_1 \in X$ and taking $s_j^{(n,m)} = \frac{1}{j(j+1)}$, $j = n, \dots, m-1$, we have, for $t > 0$ and $m, n \in \mathbb{N}$ with $m > n$, $\max_{n \leq j \leq m-1} \frac{\lambda^{j-1}}{s_j^{(n,m)}} = \max_{n \leq j \leq m-1} j(j+1)\lambda^{j-1}$. We study the function $\varphi(x) := x(x+1)\lambda^{x-1}$, whose derivative is $\varphi'(x) = (2x+1)\lambda^{x-1} + x(x+1)\log(\lambda)\lambda^{x-1} = \lambda^{x-1}\left(2x+1+x(x+1)\log(\lambda)\right)$. Since $\lambda \in (0,1)$, the quadratic function $\psi(x) := \log(\lambda)\, x^2 + (\log\lambda + 2)x + 1$ is concave and has its vertex at $x = -\frac{1}{2} - \frac{1}{\log(\lambda)}$, which is arbitrarily large if λ is arbitrarily close to zero. However, there exists $n_1 \in \mathbb{N}$ large enough (depending just on λ) such that, for every $x > n_1$, $\psi(x) < 0$. Therefore, for every $x > n_1$, $\varphi'(x) < 0$ and, thus, φ is decreasing on $(n_1, +\infty)$. In consequence, if we take $t > 0$ and $m, n \in \mathbb{N}$ with $m > n > n_1$, then $\max_{n \leq j \leq m-1} \frac{\lambda^{j-1}}{s_j^{(n,m)}} = n(n+1)\lambda^{n-1} \to 0$, as $n \to \infty$. Hence, for $\varepsilon \in (0,1)$ and $t > 0$ fixed, there exists $n_0 \in \mathbb{N}$ with $n_0 \geq n_1$ such that, for $m, n > n_0$, with $m > n$, $\max_{n \leq j \leq m-1} \frac{\lambda^{j-1}}{s_j^{(n,m)}} < \frac{t}{d(x_1, T(x_1))} \frac{\varepsilon}{1-\varepsilon}$ and, in consequence, $*_{j=n}^{m-1} \left[1 + \mu(t s_j^{(n,m)})\lambda^j\right]^{-1} = *_{j=n}^{m-1} \left[1 + \frac{d(x_1, T(x_1))}{t} j(j+1)\lambda^{j-1}\right]^{-1} > 1 - \varepsilon$. Here, we have considered that $d(x_1, T(x_1)) > 0$ since the condition $T(x_1) = x_1$ leads to a trivial case. Therefore, since $\lambda \in (0,1)$, condition (h2*) holds for $M = M_d$ and $* = *_m$, independently of the choice of x_1.

Now, we consider the general case $k \in \mathbb{N}$. For $M = M_d$, $* = *_m$ and any $x_1, \dots, x_k \in X$, we get, for $t > 0$ and $m, n \in \mathbb{N}$ with $m > n$, taking $s_j^{(n,m)} = \frac{1}{j(j+1)}$, $j = n, \dots, m-1$, that

$$\max_{n \leq j \leq m-1} \frac{\lambda^{\frac{j}{k}}}{s_j^{(n,m)}} = \max\left\{ n(n+1)\lambda^{\frac{n}{k}}, \dots, (m-2)(m-1)\lambda^{\frac{m-2}{k}}, (m-1)m\lambda^{\frac{m-1}{k}} \right\}.$$

We consider the function $\widetilde{\varphi}(x) := x(x+1)\nu^x$, being $\nu = \lambda^{\frac{1}{k}}$, where $\widetilde{\varphi}'(x) = \nu^x\left(2x+1+x(x+1)\log(\nu)\right)$. The sign of $\widetilde{\varphi}'$ coincides with the sign of the function $\widetilde{\psi}$, given by $\widetilde{\psi}(x) := \log(\nu)\, x^2 + (\log\nu + 2)x + 1$. Since $\lambda \in (0,1)$, also $\nu \in (0,1)$ and the graph of $\widetilde{\psi}$ is a concave parabola with vertex at $x = -\frac{1}{2} - \frac{1}{\log(\nu)}$.

Similarly to the case $k=1$, there exists $\widetilde{n_1}\in\mathbb{N}$ large enough (depending on λ) such that, for every $x>\widetilde{n_1}$, $\widetilde{\psi}(x)<0$, hence, for every $x>\widetilde{n_1}$, $\widetilde{\varphi}'(x)<0$ and $\widetilde{\varphi}$ is decreasing on $(\widetilde{n_1},+\infty)$. Therefore, for fixed $t>0$ and taking $m,\, n\in\mathbb{N}$ with $m>n>\widetilde{n_1}$, we get $\max_{n\le j\le m-1}\frac{\lambda^{j-1}}{s_j^{(n,m)}}=n(n+1)\lambda^{\frac{n}{k}}\to 0,$ as $n\to\infty$. This proves that, for $\varepsilon\in(0,1)$ and $t>0$ fixed, there exists $n_0\in\mathbb{N}$ with $n_0\ge\widetilde{n_1}$ such that, for $m,\, n>n_0$, with $m>n$,

$$\max_{n\le j\le m-1}\frac{\lambda^{\frac{j}{k}}}{s_j^{(n,m)}}<\frac{t}{\max\left\{\max\limits_{1\le i\le k-1}\frac{d(x_i,x_{i+1})}{\lambda^{\frac{i}{k}}},\frac{d(x_k,T(x_1,\dots,x_k))}{\lambda}\right\}}\frac{\varepsilon}{1-\varepsilon}.$$

Since $\lambda>0$ and $s_j^{(n,m)}>0$, for every $j=n,\dots,m-1$, we have, following the calculations in Remark 4, for $t>0$ and $m,\, n\in\mathbb{N}$ with $m>n$, that

$$\mathop{*}_{j=n}^{m-1}\left[1+\mu(ts_j^{(n,m)})\lambda^{\frac{j}{k}}\right]^{-1}\ge\left[1+\frac{1}{t}\right.$$
$$\left.\times\max\left\{\max_{1\le i\le k-1}\frac{d(x_i,x_{i+1})}{\lambda^{\frac{i}{k}}},\ \frac{d(x_k,T(x_1,\dots,x_k))}{\lambda}\right\}\max_{n\le j\le m-1}\frac{\lambda^{\frac{j}{k}}}{s_j^{(n,m)}}\right]^{-1}.$$

Hence, we have proved that, for $\varepsilon\in(0,1)$ and $t>0$ fixed, there exists $n_0\in\mathbb{N}$ such that, for $m,\, n>n_0$ with $m>n$, $*_{j=n}^{m-1}\left[1+\mu(ts_j^{(n,m)})\lambda^{\frac{j}{k}}\right]^{-1}>1-\varepsilon$. In the previous inequalities, we have assumed that

$$\max\left\{\max_{1\le i\le k-1}\frac{d(x_i,x_{i+1})}{\lambda^{\frac{i}{k}}},\frac{d(x_k,T(x_1,\dots,x_k))}{\lambda}\right\}>0,$$

since the opposite situation leads to a trivial case.

Again, since $\lambda\in(0,1)$, condition (H2*) holds for $M=M_d$ and $*=*_m$, independently of the choice of $x_1,\dots,x_k$.

Corollary 4. Let $(X,M,*)$ be an M-complete fuzzy metric space, k an integer with $k\ge 2$ and $T\colon X^k\to X$ a fuzzy-Prešić-Ćirić operator. Suppose that one of the conditions (H1*) or (H2*) holds. Then T has a fixed point in X. If, in addition, we suppose that

$$\text{for each } u,\, v\in X \text{ fixed with } u\ne v, \text{ there exists } t>0 \text{ such that}$$
$$\mathop{*}_{i=1}^{k}\left[\lambda\Big(\frac{1}{z_i}-1\Big)+1\right]^{-1}>M(u,v,t),\tag{2.12}$$

where $z_i=M(u,v,t\,r_i)$, for $i=1,\dots,k$, for some sequence of values $r_i>0$, $i=1,\dots,k$, with $\sum_{i=1}^{k}r_i\le 1$, then T has a unique fixed point.

Proof. The existence of fixed points follows from the proof of Theorem 3. For the uniqueness of fixed point, we suppose that $u,\, v\in X$ are fixed points of T with $u\ne v$. Then, using the nondecreasing character of $M(x,y,\cdot)$, for all $x,\, y\in X$, we have, for any $t>0$,

$$M(u,v,t)=M(T(u,\dots,u),T(v,\dots,v),t)\ge M(T(u,\dots,u),T(u,\dots,u,v),t\,r_1)$$

$$* M(T(u,\dots,u,v),T(u,\dots,u,v,v),t\,r_2)*\cdots* M(T(u,u,v,\dots,v), T(u,v,\dots,v),t\,r_{k-1})* M(T(u,v,\dots,v),T(v,\dots,v),t\,r_k). \tag{2.13}$$

Similarly to the proof of Corollary 2, using (2.2), we get, for every $t>0$, $M(T(u,\dots,u),T(u,\dots,u,v),t\,r_1)\ge\left[\lambda\left(\frac{1}{M(u,v,t\,r_1)}-1\right)+1\right]^{-1}$ and, similarly, $M(T(u,\dots,u,v),T(u,\dots,u,v,v),t\,r_2)\ge\left[\lambda\left(\frac{1}{M(u,v,t\,r_2)}-1\right)+1\right]^{-1},\dots,$ $M(T(u,u,v,\dots,v),T(u,v,\dots,v),t\,r_{k-1})\ge\left[\lambda\left(\frac{1}{M(u,v,t\,r_{k-1})}-1\right)+1\right]^{-1}$ $M(T(u,v,\dots,v),T(v,\dots,v),t\,r_k)\ge\left[\lambda\left(\frac{1}{M(u,v,t\,r_k)}-1\right)+1\right]^{-1}$.

Therefore, the previous inequalities and (2.13) imply, for every $t>0$, that $M(u,v,t)\ge *_{i=1}^{k}\left[\lambda\left(\frac{1}{M(u,v,t\,r_i)}-1\right)+1\right]^{-1}$. Hence, from (2.12), there exists $t>0$ such that $M(u,v,t)>M(u,v,t)$ and we obtain a contradiction again, so that the fixed point of T is unique. □

Remark 7. Concerning condition (2.12), we note that the expression of the values r_i, $i=1,\dots,k$, can be of similar type to $s_j^{(n,m)}$ in (H2*) or different, provided that the requirements $r_i>0$, $i=1,\dots,k$, and $\sum_{i=1}^{k} r_i\le 1$ are fulfilled.

Remark 8. As a final remark concerning the fixed point results, instead of $(X,M,*)$ an M-complete fuzzy metric space, we consider the hypothesis that $(X,M,*)$ is a G-complete fuzzy metric space. Then we can remove the restrictions (H1), (H2) in Theorem 2 and Corollary 2 and (h1), (h2) in Corollary 1. This comes from the proof of Theorem 2, in this case we can start with arbitrary points $x_1,\dots,x_k$ in X and, to prove that the sequence defined is G-Cauchy, we just note that, for $t>0$ and $p\in\mathbb{N}$ fixed, we have

$$\begin{aligned} M(x_n,x_{n+p},t) &\ge \mathop{*}_{j=n}^{n+p-1} M(x_j,x_{j+1},ts_j^{(n)}) \\ &= \mathop{*}_{j=n}^{n+p-1} M_j(ts_j^{(n)}) \ge \mathop{*}_{j=n}^{n+p-1}\left[1+\mu(ts_j^{(n)})\theta^n\right]^{-1}, \end{aligned}$$

where $s_j^{(n)}=\frac{1}{2^{j-n+1}}$, $j=n,\dots,n+p-1$.

Note that the last term in the previous inequality consists of a fixed number of terms (for every n), that is, p terms, each of one tends to 1 as $n\to\infty$ due to $\theta\in(0,1)$ and the fact that $s_j^{(n)}$ represents a constant sequence for each j fixed, in the sense that $s_n^{(n)}=\frac{1}{2}$, for every n, $s_{n+1}^{(n)}=\frac{1}{2^2}$, for every n, $\dots$, $s_{n+p-1}^{(n)}=\frac{1}{2^p}$, for every n. Hence $\lim_{n\to\infty} M(x_n,x_{n+p},t)=1*\overset{p)}{\cdots}*1=1$, for each $t>0$ and $p>0$, and $\{x_n\}$ is G-Cauchy.

3 Some properties of fuzzy contractive sequences

We include some conclusions on fuzzy contractive sequences that are derived from the proof of the main results in the previous section.

Definition 6. Let $(X, M, *)$ be a fuzzy metric space and k a positive integer. We say that $\{x_n\} \subset X$ is a fuzzy contractive sequence if there exists $\lambda \in (0,1)$ such that

$$\frac{1}{M(x_{n+k}, x_{n+k+1}, t)} - 1 \leq \lambda \max_{1 \leq i \leq k} \left\{ \frac{1}{M(x_{n+i-1}, x_{n+i}, t)} - 1 \right\}, \tag{3.1}$$

for all $t > 0$ and $n \in \mathbb{N}$. Condition (3.1) can also be written as $M(x_{n+k}, x_{n+k+1}, t) \geq \left[\lambda \max_{1\leq i\leq k} \left\{ \frac{1}{M(x_{n+i-1}, x_{n+i}, t)} - 1 \right\} + 1\right]^{-1}$, for all $t > 0$ and $n \in \mathbb{N}$, where $\lambda \in (0,1)$.

This notion is a generalization of Definition 3.8 [9] since, for $k = 1$, it is reduced to $\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq \lambda \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1 \right)$, for all $t > 0$ and $n \in \mathbb{N}$, where $\lambda \in (0,1)$.

For the case $k = 1$, it is proposed in [9] the following open question: Is a fuzzy contractive sequence a Cauchy sequence in George and Veeramani's sense (that is, an M-Cauchy sequence). We study this problem for an arbitrary $k \in \mathbb{N}$, by imposing sufficient conditions which guarantee the validity of this assertion.

For a given sequence $\{x_n\}$, consider $\mu(z) := \max_{1\leq i\leq k} \frac{1}{\lambda^{\frac{i}{k}}} \left[\frac{1}{M(x_i, x_{i+1}, z)} - 1 \right]$ and the hypotheses:

(HS1) $* \in \mathcal{H}$ and $\inf_{t>0} M(x_i, x_{i+1}, t) > 0$, for all $i = 1, 2, \ldots, k$.

(HS2) For each $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for $m, n > n_0$, with $m > n$, we get $*_{j=n}^{m-1} \left[1 + \mu(t s_j^{(n,m)}) \lambda^{\frac{j}{k}} \right]^{-1} > 1 - \varepsilon$, for some collection of values $s_j^{(n,m)} > 0$, $j = n, \ldots, m-1$, such that $\sum_{j=n}^{m-1} s_j^{(n,m)} \leq 1$.

Theorem 4. *Let $(X, M, *)$ be a fuzzy metric space and k a positive integer. Let $\{x_n\} \subset X$ be a fuzzy contractive sequence. Suppose that one of the conditions (HS1) or (HS2) holds. Then $\{x_n\}$ is an M-Cauchy sequence.*

Proof. As in the proof of Theorem 2, we denote $M_n(t) := M(x_n, x_{n+1}, t)$, for $n \in \mathbb{N}$ and $t > 0$ and $\mu(t) = \max_{1\leq i\leq k} \left\{ \frac{1}{\theta^i} \left[\frac{1}{M_i(t)} - 1 \right] \right\}$, where $\theta = \lambda^{\frac{1}{k}}$. Similarly to the proof of Theorem 2, by induction, we prove that

$$1/M_n(t) - 1 \leq \mu(t)\theta^n, \qquad \forall\, n \in \mathbb{N}, \ \forall\, t > 0. \tag{3.2}$$

Indeed, it is true for $n = 1, 2, \ldots, k$. Assuming that it is true for $n, n+1, \ldots, n+k-1$, we have, from (3.1),

$$\frac{1}{M_{n+k}(t)} - 1 = \frac{1}{M(x_{n+k}, x_{n+k+1}, t)} - 1 \leq \lambda \max_{1\leq i\leq k} \left\{ \frac{1}{M(x_{n+i-1}, x_{n+i}, t)} - 1 \right\}$$
$$= \lambda \max_{1\leq i\leq k} \left\{ \frac{1}{M_{n+i-1}(t)} - 1 \right\} \leq \lambda \max_{1\leq i\leq k} \left\{ \mu(t)\, \theta^{n+i-1} \right\} \leq \lambda \mu(t) \theta^n = \mu(t) \theta^{n+k},$$

for $t > 0$, where we have used that $\theta = \lambda^{\frac{1}{k}} < 1$.

To check that $\{x_n\}$ is an M-Cauchy sequence, we take $\varepsilon \in (0,1)$ and $t > 0$ fixed. Then, by (3.2), using the nondecreasing character of $M(x,y,\cdot)$ for every x, $y \in X$ and following the proof of Theorem 4.8 [9], we have, for $n, m \in \mathbb{N}$ with $m > n$, that $M(x_n, x_m, t) \geq *_{j=n}^{m-1} M_j(ts_j^{(n,m)}) \geq *_{j=n}^{m-1} \left[1 + \mu(t\, s_j^{(n,m)})\theta^j\right]^{-1}$, for any collection of values $s_j^{(n,m)} > 0$, $j = n, \ldots, m-1$, with $\sum_{j=n}^{m-1} s_j^{(n,m)} \leq 1$.

If (HS1) holds, then $\mu := \sup\limits_{t>0} \mu(t) \in [0, \infty)$, therefore, for $n, m \in \mathbb{N}$ with $m > n$, we get $M(x_n, x_m, t) \geq *^{m-n} \left[1 + \mu\theta^n\right]^{-1}$. Since $* \in \mathcal{H}$, the proof is complete similarly to the proof of Theorem 2.

On the other hand, (HS2) provides trivially the character of M-Cauchy sequence for $\{x_n\}$. $\square$

Remark 9. If $M = M_d$, condition (HS1) is satisfied only for constant sequences $\{x_n\}$.

We conclude the paper with some applications of our Theorem 3 (and Corollary 4) to certain nonlinear differential equations subject to initial conditions.

4 Applications to differential equations

In this section, we study the initial value problem for some classes of second order differential equations. First, we consider the autonomous case, as follows.

Let $T > 0$, $I = [0, T]$ and consider the problem:

$$x''(t) = \xi(x(t), x(t), \ldots, x(t)),\ t \in I, \qquad x(0) = \alpha,\ x'(0) = \beta, \tag{4.1}$$

where α, $\beta \in \mathbb{R}$ and $\xi \colon \mathbb{R}^k = \mathbb{R} \times \overset{(k)}{\cdots} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
The Green's function associated with (4.1) is given by $G(t,\tau) = t - \tau$ for $t > \tau$, and $G(t,\tau) = 0$ for $0 \leq t \leq \tau$, in such a way that, as it can be easily seen, the solution to (4.1) is given by the solution of the integral equation of the following form:

$$\begin{aligned} x(t) &= \int_0^T G(t,\tau)\xi(x(\tau), x(\tau), \ldots, x(\tau))d\tau + \zeta(t) \\ &= \int_0^t (t-\tau)\xi(x(\tau), x(\tau), \ldots, x(\tau))d\tau + \zeta(t),\ \text{for } t \in I, \end{aligned} \tag{4.2}$$

where $\zeta(t) = \alpha + \beta t$.

To define the concept of solution to (4.1), we consider $C^2(I, \mathbb{R})$, the space of all functions from I into $\mathbb{R}$ having continuous second order derivative on I. A solution to (4.1) is a function $x \in C^2(I, \mathbb{R})$ which satisfies the conditions in (4.1). The procedure we follow to prove the existence of solutions to problem (4.1) is to establish a connection between them and the solutions to the integral equation (4.2).

To study the existence of solutions to the integral equation (4.2), we consider $C(I, \mathbb{R})$, the Banach space of all continuous functions from $I = [0, T]$ into

$\mathbb{R}$, endowed with the supremum norm, defined as: $\|x\|_\infty := \sup_{t\in I} \mid x(t) \mid$, $x \in C(I,\mathbb{R})$. Notice that $C(I,\mathbb{R})$ is also a Banach space with the Bielecki norm given by $\|x\|_B := \sup_{t\in I} \left\{\mid x(t) \mid e^{-bt}\right\}$, $x \in C(I,\mathbb{R})$, where $b > 0$ is arbitrary but fixed. It is easy to see that the two norms $\|\cdot\|_\infty$ and $\|\cdot\|_B$ are equivalent on $I = [0,T]$. For our purpose, we use the Bielecki norm instead of the supremum norm. The Bielecki metric induced by the Bielecki norm is given by the expression $d_B(x,y) := \sup_{t\in I} \left\{\mid x(t) - y(t) \mid e^{-bt}\right\}$, x, $y \in C(I,\mathbb{R})$. The standard fuzzy metric $M_{d_B} \colon [C(I,\mathbb{R})]^2 \times (0,\infty) \to [0,1]$ induced by d_B is defined as: $M_{d_B}(x,y,a) = \dfrac{a}{a + d_B(x,y)}, \quad \forall\, x, y \in C(I,\mathbb{R}),\ \forall a > 0.$
Then, it is easy to see that the min-fuzzy metric space $(C(I,\mathbb{R}), M_{d_B}, *_m)$ is an M-complete fuzzy metric space. Define an operator Φ: $[C(I,\mathbb{R})]^k \to C(I,\mathbb{R})$ by $[\Phi(x_1,x_2,\ldots,x_k)](t) = \int_0^t (t-\tau)\xi(x_1(\tau),x_2(\tau),\ldots,x_k(\tau))d\tau + \zeta(t)$, for $t \in I$ and $x_1, \ldots, x_k \in C(I,\mathbb{R})$.
It is obvious that the solutions to the integral equation (4.2) coincide with the fixed points of the operator Φ, i.e., $x \in C(I,\mathbb{R})$ such that $[\Phi(x,x,\ldots,x)](t) = x(t)$, for every $t \in I$. Moreover, a solution to (4.1) trivially satisfies the integral equation (4.2) and, if $x \in C(I,\mathbb{R})$ is a solution to the integral equation (4.2), then we can prove that $x \in C^2(I,\mathbb{R})$ and the conditions in (4.1) are fulfilled. Hence, the solutions to the initial value problem (4.1) are the fixed points of the operator Φ. All these considerations allow us to prove the existence and uniqueness of solution to the initial value problem (4.1), as established in the following theorem.

Theorem 5. *Let k be a positive integer and suppose that the following conditions are satisfied:*

(a) $\xi \colon \mathbb{R}^k \to \mathbb{R}$ is a continuous function;

(b) there exists $L > 0$ such that, for all $z_1, z_2, \ldots, z_k, z_{k+1} \in \mathbb{R}$, we have $\mid \xi(z_1,\ldots,z_k) - \xi(z_2,\ldots,z_{k+1}) \mid \leq L \max_{1\leq i\leq k} \mid z_i - z_{i+1} \mid$.

Then the initial value problem (4.1) *has a unique solution.*

Proof. We consider $(C(I,\mathbb{R}), M_{d_B}, *_m)$ the min-fuzzy metric space induced by the Bielecki metric d_B on $C(I,\mathbb{R})$. Since $b > 0$ can be selected arbitrarily, we choose $b = kLT > 0$ then, for all $x_1,\ldots,x_k,x_{k+1} \in C(I,\mathbb{R})$, we have

$$\begin{aligned}
&d_B(\Phi(x_1,\ldots,x_k),\Phi(x_2,\ldots,x_{k+1}))\\
&\qquad= \sup_{t\in I}\left|\int_0^t (t-\tau)[\xi(x_1(\tau),\ldots,x_k(\tau)) - \xi(x_2(\tau),\ldots,x_{k+1}(\tau))]e^{-bt}d\tau\right|\\
&\qquad\leq \sup_{t\in I}\int_0^t (t-\tau) \mid \xi(x_1(\tau),\ldots,x_k(\tau)) - \xi(x_2(\tau),\ldots,x_{k+1}(\tau)) \mid e^{-bt}d\tau\\
&\qquad\leq \sup_{t\in I}\int_0^t (t-\tau)\, L \max_{1\leq i\leq k}\{\mid x_i(\tau) - x_{i+1}(\tau) \mid\}\, e^{-b\tau}e^{b(\tau-t)}d\tau\\
&\qquad\leq L \max_{1\leq i\leq k}\{d_B(x_i,x_{i+1})\} \sup_{t\in I}\left\{e^{-bt}\int_0^t (t-\tau)e^{b\tau}d\tau\right\}
\end{aligned}$$

$$\leq L \max_{1\leq i\leq k}\{d_B(x_i,x_{i+1})\} \sup_{t\in I}\left\{te^{-bt}\int_0^t e^{b\tau}d\tau\right\}$$
$$\leq \frac{LT}{b}(1-e^{-bT}) \max_{1\leq i\leq k}\{d_B(x_i,x_{i+1})\}.$$

Since $b = kLT$, we have, for all $x_1,\ldots,x_k,x_{k+1} \in C(I,\mathbb{R})$,

$$d_B(\Phi(x_1,\ldots,x_k),\Phi(x_2,\ldots,x_{k+1})) \leq \lambda \max_{1\leq i\leq k}\{d_B(x_i,x_{i+1})\},$$

where $0 < \lambda = \frac{LT}{b}(1-e^{-bT}) = \frac{1}{k}(1-e^{-bT}) < 1$. Therefore,

$$M_{d_B}(\Phi(x_1,\ldots,x_k),\Phi(x_2,\ldots,x_{k+1}),a)=\frac{a}{a+d_B(\Phi(x_1,\ldots,x_k),\Phi(x_2,\ldots,x_{k+1}))}$$
$$\geq \frac{a}{a+\lambda \max\limits_{1\leq i\leq k}\{d_B(x_i,x_{i+1})\}} = \left[\lambda \max_{1\leq i\leq k}\left\{\frac{1}{M_{d_B}(x_i,x_{i+1},a)}-1\right\}+1\right]^{-1},$$

for all $x_1, x_2, \ldots, x_{k+1} \in C(I,\mathbb{R})$ and $a > 0$. Thus, Φ is a fuzzy-Prešić-Ćirić operator. Besides, it is clear that condition (H2*) holds for arbitrary choices of $x_1,\ldots, x_k \in C(I,\mathbb{R})$, since $\lambda \in (0,1)$, taking $s_j = s_j^{(n,m)} = \frac{1}{j(j+1)}$, for $j = n,\ldots,m-1$ (see Remark 6). Hence, by Theorem 3, Φ has a fixed point in $C(I,\mathbb{R})$, which gives a solution to (4.1). Furthermore, for x, $y \in C(I,\mathbb{R})$, we have, by (b),

$$\begin{aligned}
&d_B(\Phi(x,\ldots,x),\Phi(y,\ldots,y))\\
&\quad= \sup_{t\in I}\left|\int_0^t (t-\tau)[\xi(x(\tau),\ldots,x(\tau)) - \xi(y(\tau),\ldots,y(\tau))]e^{-bt}d\tau\right|\\
&\quad\leq \sup_{t\in I}\int_0^t (t-\tau)\Big[\,|\,\xi(x(\tau),\ldots,x(\tau)) - \xi(x(\tau),\ldots,x(\tau),y(\tau))\,|\\
&\qquad\quad + \cdots + |\,\xi(x(\tau),y(\tau),\ldots,y(\tau)) - \xi(y(\tau),\ldots,y(\tau))\,|\,\Big]e^{-bt}d\tau\\
&\quad\leq \sup_{t\in I}\int_0^t (t-\tau)\,k\,L\,|\,x(\tau)-y(\tau)\,|\,e^{-bt}d\tau\\
&\quad\leq kL\,d_B(x,y)\sup_{t\in I}\left\{te^{-bt}\int_0^t e^{b\tau}d\tau\right\} \leq \frac{kLT}{b}(1-e^{-bT})\,d_B(x,y).
\end{aligned}$$

Since $b = kLT$, we have

$$d_B(\Phi(x,\ldots,x),\Phi(y,\ldots,y)) \leq (1-e^{-bT})d_B(x,y) < d_B(x,y),$$

for all x, $y \in C(I,\mathbb{R})$ with $x \neq y$. Therefore, by the definition of M_{d_B},

$$M_{d_B}(\Phi(x,\ldots,x),\Phi(y,\ldots,y),a) > M_{d_B}(x,y,a),\ \forall\, a > 0. \tag{4.3}$$

Thus, all the conditions of Theorem 3 are satisfied and, in consequence, Φ has a unique fixed point in $C(I,\mathbb{R})$, which is the unique solution to the initial value problem (4.1). $\square$

Remark 10. Note that, in the proof of Theorem 5, if $k \geq 2$, an alternative way to check the validity of (4.3) in order to achieve the uniqueness of solutions is to use the ideas in Corollary 4, since, with the selection of the minimum t-norm $*_m$ and, for example, $r_i = \frac{1}{2^i}$, $i = 1, \ldots, k$, condition (2.12) is reduced to the following one: for each $u, v \in X$ fixed with $u \neq v$, there exists $t > 0$ such that $\left[\lambda\left(\frac{1}{M(u,v,t/2^i)} - 1\right) + 1\right]^{-1} > M(u, v, t)$, for every $i = 1, \ldots, k-1$.

To check its validity, we take $u, v \in X$ fixed with $u \neq v$. For $M = M_{d_B}$ and $a > 0$, we have

$$\begin{aligned}\left[\lambda\left(\frac{1}{M_{d_B}(u,v,a/2^i)} - 1\right) + 1\right]^{-1} &= \left[\lambda\left(\frac{a/2^i + d_B(u,v)}{a/2^i} - 1\right) + 1\right]^{-1} \\ = \left[\lambda\frac{d_B(u,v)}{a/2^i} + 1\right]^{-1} &= \frac{a/2^i}{\lambda d_B(u,v) + a/2^i} = \frac{a}{\lambda 2^i d_B(u,v) + a},\end{aligned}$$

so that, since $d_B(u, v) > 0$, this expression is greater than $M_{d_B}(u, v, a)$, for every $i = 1, \ldots, k-1$, if and only if $\lambda \cdot 2^i < 1$, for every $i = 1, \ldots, k-1$, that is, $\lambda < 2^{1-k}$. Note that λ is taken as $\lambda = \dfrac{LT}{b}(1-e^{-bT})$ in the proof of Theorem 5. Thus, in order to prove that Φ is a fuzzy-Prešić-Ćirić operator, we just have to choose $b > 0$ such that $b > LT$. Now, for the validity of (2.12), it suffices to choose $b > 0$ with $\dfrac{LT2^{k-1}}{b}(1 - e^{-bT}) < 1$, that is, $b > LT2^{k-1}$. This provides uniqueness of solution for $k \geq 2$.

Instead of problem (4.1), we could have considered a non-autonomous problem of the type

$$x''(t) = \xi(t, x(t), x(t), \ldots, x(t)),\ t \in I = [0, T], \quad x(0) = \alpha,\ x'(0) = \beta, \quad (4.4)$$

where α, $\beta \in \mathbb{R}$ and $\xi \colon I \times \mathbb{R}^k \to \mathbb{R}$ is continuous.

Taking $\zeta(t) = \alpha + \beta t$, $t \in I$, it is clear that the solutions to (4.4) coincide with those of the integral equation

$$\begin{aligned} x(t) &= \int_0^T G(t,\tau)\xi(\tau, x(\tau), x(\tau), \ldots, x(\tau))d\tau + \zeta(t) \\ &= \int_0^t (t-\tau)\xi(\tau, x(\tau), x(\tau), \ldots, x(\tau))d\tau + \zeta(t), \text{ for } t \in I \end{aligned}$$

and also with the fixed points of the mapping $\widetilde{\Phi} \colon [C(I, \mathbb{R})]^k \to C(I, \mathbb{R})$ defined as $[\widetilde{\Phi}(x_1, x_2, \ldots, x_k)](t) = \int_0^t (t-\tau)\xi(\tau, x_1(\tau), x_2(\tau), \ldots, x_k(\tau))d\tau + \zeta(t)$, for $t \in I$ and $x_1, \ldots, x_k \in C(I, \mathbb{R})$. Thus, the following extension of Theorem 5 follows.

Theorem 6. *Let k be a positive integer and suppose that the following conditions are satisfied:*

(a) $\xi \colon I \times \mathbb{R}^k \to \mathbb{R}$ is a continuous function;

(b) there exists a nonnegative and integrable function $L : I \to \mathbb{R}$ such that, for all $t \in I$ and $z_1, z_2, \ldots, z_k, z_{k+1} \in \mathbb{R}$, we have

$$| \xi(t, z_1, \ldots, z_k) - \xi(t, z_2, \ldots, z_{k+1}) | \leq L(t) \max_{1\leq i\leq k} | z_i - z_{i+1} |$$

and there exists $b > 0$ such that

$$k \sup_{t\in I} \left\{ e^{-bt} \int_0^t (t-\tau)L(\tau)e^{b\tau}\, d\tau \right\} < 1. \tag{4.5}$$

Then the initial value problem (4.4) *has a unique solution.*

Proof. We consider again $(C(I,\mathbb{R}), M_{d_B}, *_m)$ the min-fuzzy metric space induced by the Bielecki metric d_B on $C(I,\mathbb{R})$, where $b > 0$ is given by the statement. Hence, similarly to the proof of Theorem 5, we have, for all $x_1, \ldots, x_k, x_{k+1} \in C(I,\mathbb{R})$,

$$\begin{aligned}
&d_B(\widetilde{\Phi}(x_1, \ldots, x_k), \widetilde{\Phi}(x_2, \ldots, x_{k+1})) \\
&\leq \sup_{t\in I} \int_0^t (t-\tau)\, L(\tau) \max_{1\leq i\leq k} \{| x_i(\tau) - x_{i+1}(\tau) |\}\, e^{-b\tau} e^{b(\tau-t)} d\tau \\
&\leq \lambda \max_{1\leq i\leq k} \{d_B(x_i, x_{i+1})\},
\end{aligned}$$

where $0 < \lambda := \sup_{t\in I} \left\{ e^{-bt} \int_0^t (t-\tau)L(\tau)e^{b\tau} d\tau \right\} < \frac{1}{k} \leq 1$.

Besides, if $x, y \in C(I,\mathbb{R})$, by (b),

$$\begin{aligned}
&d_B(\widetilde{\Phi}(x, \ldots, x), \widetilde{\Phi}(y, \ldots, y)) \\
&\quad \leq \sup_{t\in I} \int_0^t (t-\tau) \Big[L(\tau) \mid x(\tau) - y(\tau) \mid + \cdots + L(\tau) \mid x(\tau) - y(\tau) \mid \Big] e^{-bt} d\tau \\
&\quad = k \sup_{t\in I} \int_0^t (t-\tau)\, L(\tau) \mid x(\tau) - y(\tau) \mid e^{-b\tau} e^{b(\tau-t)} d\tau \leq k\, d_B(x,y)\lambda,
\end{aligned}$$

so that $d_B(\widetilde{\Phi}(x, \ldots, x), \widetilde{\Phi}(y, \ldots, y)) < d_B(x,y)$, for all $x, y \in C(I,\mathbb{R})$ with $x \neq y$ ($d_B(x,y) > 0$). Since (H2*) also holds, then Theorem 3 applies. □

Remark 11. Condition (4.5) trivially holds if there exists $b > 0$ such that $\sup_{t\in I} \left\{ e^{-bt} \int_0^t L(\tau)e^{b\tau} d\tau \right\} < \frac{1}{kT}$, or if $\lim_{b\to\infty} \left(\sup_{t\in I} \left\{ e^{-bt} \int_0^t L(\tau)e^{b\tau} d\tau \right\} \right) < \frac{1}{kT}$, since $k \sup_{t\in I} \left\{ e^{-bt} \int_0^t (t-\tau)L(\tau)e^{b\tau}\, d\tau \right\} \leq kT \sup_{t\in I} \left\{ e^{-bt} \int_0^t L(\tau)e^{b\tau}\, d\tau \right\}$. In particular, if L is nonnegative, integrable and bounded (there exists $\mathcal{L} > 0$ such that $L(t) \leq \mathcal{L}$, $t \in I$), then

$$k \sup_{t\in I} \left\{ e^{-bt} \int_0^t (t-\tau)L(\tau)e^{b\tau} d\tau \right\} \leq kT\mathcal{L} \sup_{t\in I} \left\{ e^{-bt} \int_0^t e^{b\tau} d\tau \right\} = kT\mathcal{L} \frac{1-e^{-bT}}{b},$$

so that it is enough to choose $b > kT\mathcal{L}$.

Finally, we consider an impulsive problem of the type

$$\begin{cases} x''(t) = \xi(t, x(t), x(t), \dots, x(t)),\ t \in I = [0,T],\ t \neq t_j,\ j = 1, \dots, l, \\ x(t_j^+) = \alpha_j, \quad x'(t_j^+) = \beta_j, \quad j = 0, \dots, l, \end{cases} \tag{4.6}$$

where α_j, $\beta_j \in \mathbb{R}$, $j = 0, \dots, l$, $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = T$ and $\xi\colon I \times \mathbb{R}^k \to \mathbb{R}$ is such that, for every $j = 0, \dots, l$, its restriction to $(t_j, t_{j+1}] \times \mathbb{R}^k$ is continuous and admits a continuous extension to the set $[t_j, t_{j+1}] \times \mathbb{R}^k$.

To define the concept of solution to problem (4.6), we consider the spaces

$$\begin{aligned} PC(I,\mathbb{R}) &= \{x : I \to \mathbb{R} \,:\, x \text{ is continuous in } I \setminus \{t_1, \dots, t_l\} \\ &\qquad \text{and } \exists\, x(t_j^+),\ x(t_j^-) = x(t_j),\ j = 1, \dots, l\} \\ &= \{x : I \to \mathbb{R} \,: x \in C((t_j, t_{j+1}), \mathbb{R}),\ j = 0, \dots, l, \text{and } \exists\, x(0^+) = x(0), \\ &\qquad x(T^-) = x(T),\ x(t_j^+),\ x(t_j^-) = x(t_j),\ j = 1, \dots, l\}, \\ E &:= \{x \in PC(I,\mathbb{R}) \,: x \in C^2(I \setminus \{t_1, \dots, t_l\}, \mathbb{R}) \\ &\qquad \text{and } \exists\, x'(t_j^+),\ x'(t_j^-),\ x''(t_j^+),\ x''(t_j^-),\ j = 1, \dots, l\} \\ &= \{x \in PC(I,\mathbb{R}) \,: x \in C^2((t_j, t_{j+1}), \mathbb{R}),\ j = 0, \dots, l, \text{ and } \exists\, x'(0^+) = x'(0), \\ &\qquad x''(0^+) = x''(0),\ x'(T^-) = x'(T),\ x''(T^-) = x''(T), \\ &\qquad x'(t_j^+),\ x'(t_j^-),\ x''(t_j^+),\ x''(t_j^-),\ j = 1, \dots, l\}. \end{aligned}$$

Hence, a solution to (4.6) is a function $x \in E$ satisfying the conditions in (4.6). For this problem (4.6), the Green's function $G : I \times I \to \mathbb{R}$ is given by

$$G(t,\tau) = \begin{cases} t - \tau, & t_j < \tau < t \le t_{j+1}, \text{ for some } j = 0, \dots, l, \\ 0, & \text{otherwise.} \end{cases}$$

We can also use the functions $G_j : [t_j, t_{j+1}] \times [t_j, t_{j+1}] \to \mathbb{R}$, $j = 0, \dots, l$, defined by $G_j(t,\tau) = t - \tau$, if $t_j \le \tau < t \le t_{j+1}$, and $G_j(t,\tau) = 0$, if $t_j \le t \le \tau \le t_{j+1}$, in such a way that $G(t,\tau) = G_j(t,\tau)$, for $(t,\tau) \in (t_j, t_{j+1}) \times (t_j, t_{j+1})$. Therefore, taking $\zeta_j(t) = \alpha_j + \beta_j(t - t_j)$, for $t \in (t_j, t_{j+1}]$, and $j = 0, \dots, l$, the solutions to (4.6) are the solutions $x \in PC(I,\mathbb{R})$ to the family of integral equations:

$$\begin{aligned} x(t) &= \int_0^T G(t,\tau)\xi(\tau, x(\tau), x(\tau), \dots, x(\tau))d\tau + \zeta_j(t) \\ &= \int_{t_j}^{t_{j+1}} G_j(t,\tau)\xi(\tau, x(\tau), x(\tau), \dots, x(\tau))d\tau + \zeta_j(t) \\ &= \int_{t_j}^{t} (t-\tau)\xi(\tau, x(\tau), x(\tau), \dots, x(\tau))d\tau + \zeta_j(t),\ t \in (t_j, t_{j+1}]. \end{aligned}$$

The space $PC(I,\mathbb{R})$ is a Banach space with the supremum norm defined as $\|x\|_{PC} := \sup_{t\in I} \mid x(t) \mid = \max_{0\le j\le l} \sup_{t\in(t_j,t_{j+1}]} \mid x(t) \mid$, $x \in PC(I,\mathbb{R})$ and also with the equivalent norm $\|x\|_{PCB} := \max\limits_{0\le j\le l} \sup\limits_{t\in(t_j,t_{j+1}]} \left\{\mid x(t) \mid e^{-b(t-t_j)}\right\}$, $x \in PC(I,\mathbb{R})$, where $b > 0$ is arbitrary but fixed. The distance induced by

$\|\cdot\|_{PCB}$ is $d_{PCB}(x,y) := \max_{0\le j\le l}\sup_{t\in(t_j,t_{j+1}]}\left\{\mid x(t)-y(t)\mid e^{-b(t-t_j)}\right\}$, for $x,\,y\in PC(I,\mathbb{R})$.

We consider the mapping $\widehat{\Phi}\colon [PC(I,\mathbb{R})]^k \to PC(I,\mathbb{R})$, given by

$$[\widehat{\Phi}(x_1,x_2,\ldots,x_k)](t) = \int_{t_j}^{t}(t-\tau)\xi(\tau,x_1(\tau),x_2(\tau),\ldots,x_k(\tau))d\tau + \zeta_j(t),$$

for $t\in(t_j,t_{j+1}]$, $j=0,\ldots,l$, and $x_1,\,\ldots,x_k\in PC(I,\mathbb{R})$, whose fixed points are the solutions sought.

We prove the following existence and uniqueness result for problem (4.6).

Theorem 7. *Let k be a positive integer and suppose that the following conditions are satisfied:*

(a) $\xi\colon I\times\mathbb{R}^k\to\mathbb{R}$ is such that its restriction to $(t_j,t_{j+1}]\times\mathbb{R}^k$ is continuous and admits a continuous extension to the set $[t_j,t_{j+1}]\times\mathbb{R}^k$, for $j=0,\ldots,l$.

(b) there exists a nonnegative and integrable function $L: I\to\mathbb{R}$ such that, for all $t\in I$ and $z_1,\,z_2,\,\ldots,\,z_k,\,z_{k+1}\in\mathbb{R}$, we have

$$\mid \xi(t,z_1,\ldots,z_k)-\xi(t,z_2,\ldots,z_{k+1})\mid \le L(t)\max_{1\le i\le k}\mid z_i-z_{i+1}\mid$$

and there exists $b>0$ such that

$$k\max_{0\le j\le l}\sup_{t\in(t_j,t_{j+1}]}\left\{e^{-bt}\int_{t_j}^{t}(t-\tau)L(\tau)e^{b\tau}\,d\tau\right\}<1. \tag{4.7}$$

Then the initial value problem (4.6) has a unique solution.

Proof. We take $(PC(I,\mathbb{R}),M_{d_{PCB}},*_m)$ the min-fuzzy metric space induced by the metric d_{PCB} on $PC(I,\mathbb{R})$, where $b>0$ is given in the statement. Analogously to the proof of Theorem 6, we have, for all $x_1,\ldots,x_k,x_{k+1}\in PC(I,\mathbb{R})$,

$$\begin{aligned}
&d_{PCB}(\widehat{\Phi}(x_1,\ldots,x_k),\widehat{\Phi}(x_2,\ldots,x_{k+1}))\\
&\le \max_{0\le j\le l}\sup_{t\in(t_j,t_{j+1}]}\int_{t_j}^{t}(t-\tau)\,L(\tau)\max_{1\le i\le k}\{\mid x_i(\tau)-x_{i+1}(\tau)\mid\}\,e^{-b(\tau-t_j)}e^{b(\tau-t)}d\tau\\
&\le \lambda\max_{1\le i\le k}\{d_{PCB}(x_i,x_{i+1})\},
\end{aligned}$$

where $0<\lambda:=\max_{0\le j\le l}\sup_{t\in(t_j,t_{j+1}]}\left\{e^{-bt}\int_{t_j}^{t}(t-\tau)L(\tau)e^{b\tau}d\tau\right\}<\frac{1}{k}\le 1$. Moreover, for $x,\,y\in PC(I,\mathbb{R})$, by condition (b),

$$\begin{aligned}
&d_{PCB}(\widehat{\Phi}(x,\ldots,x),\widehat{\Phi}(y,\ldots,y))\\
&\le \max_{0\le j\le l}\sup_{t\in(t_j,t_{j+1}]}\int_{t_j}^{t}(t-\tau)\,k\,L(\tau)\mid x(\tau)-y(\tau)\mid e^{-b(t-t_j)}d\tau\\
&= k\max_{0\le j\le l}\sup_{t\in(t_j,t_{j+1}]}\int_{t_j}^{t}(t-\tau)\,L(\tau)\mid x(\tau)-y(\tau)\mid e^{-b(\tau-t_j)}e^{b(\tau-t)}d\tau\\
&\le k\,d_{PCB}(x,y)\lambda,
\end{aligned}$$

in consequence, $d_{PCB}(\widehat{\Phi}(x,\ldots,x),\widehat{\Phi}(y,\ldots,y)) < d_{PCB}(x,y)$, for all $x, y \in PC(I,\mathbb{R})$ with $x \neq y$ ($d_{PCB}(x,y) > 0$). The proof is concluded by Theorem 3, due to the validity of (H2*). $\square$

Remark 12. If L is nonnegative, integrable and bounded (with upper bound $\mathcal{L} > 0$), then condition (4.7) trivially holds since

$$\begin{aligned} &k \max_{0\le j\le l} \sup_{t\in(t_j,t_{j+1}]} \left\{ e^{-bt} \int_{t_j}^{t} (t-\tau)L(\tau)e^{b\tau}\,d\tau \right\} \\ &\quad \le kT\mathcal{L} \max_{0\le j\le l} \sup_{t\in(t_j,t_{j+1}]} \left\{ e^{-bt} \int_{t_j}^{t} e^{b\tau}\,d\tau \right\} = \frac{kT\mathcal{L}}{b} \max_{0\le j\le l} \left\{ 1 - e^{-b(t_{j+1}-t_j)} \right\} \end{aligned}$$

and the same choice of $b > kT\mathcal{L}$ is useful.

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