Mathematical Modelling and Analysis
Volume 22 Number 1, January 2017, 95–105
https://doi.org/10.3846/13926292.2017.1269373

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
ISSN: 1392-6292
eISSN: 1648-3510

A Weighted Universality Theorem for Periodic Zeta-Functions

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Received June 4, 2016; revised December 2, 2016; published online January 5, 2017

Abstract. The periodic zeta-function $\zeta(s;\mathfrak{a})$, $s=\sigma+it$ is defined for $\sigma>1$ by the Dirichlet series with periodic coefficients and is meromorphically continued to the whole complex plane. It is known that the function $\zeta(s;\mathfrak{a})$, for some sequences $\mathfrak{a}$ of coefficients, is universal in the sense that its shifts $\zeta(s+i\tau;\mathfrak{a})$, $\tau\in\mathbb{R}$, approximate a wide class of analytic functions. In the paper, a weighted universality theorem for the function $\zeta(s;\mathfrak{a})$ is obtained.

Keywords: Hurwitz zeta-function, Mergelyan theorem, periodic zeta-function, universality.

AMS Subject Classification: 11M41.

1 Introduction

Let $s=\sigma+it$ be a complex variable, and $\mathfrak{a}=\{a_m : m\in\mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $k\in\mathbb{N}$. The periodic zeta-function $\zeta(s;\mathfrak{a})$ is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m}{m^s}.$$

The Hurwitz zeta-function $\zeta(s,\alpha)$ with parameter α, $0<\alpha\leqslant 1$, is given, for

$\sigma > 1$, by the series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}$$

and can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. Since, in view of periodicity of $\mathfrak{a}$,

$$\zeta(s;\mathfrak{a}) = \frac{1}{k^s} \sum_{m=1}^{k} a_m \zeta\left(s, \tfrac{m}{k}\right), \tag{1.1}$$

the periodic zeta-function also has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$\frac{1}{k} \sum_{m=1}^{k} a_m.$$

If the later quantity is equal to zero, then the function $\zeta(s;\mathfrak{a})$ is entire one.

In 1975, S.M. Voronin discovered [16] the universality of the Riemann zeta-function $\zeta(s) = \zeta(s,1)$ on the approximation of a wide class of analytic functions by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$. After Voronin's work, various authors observed that some other zeta-functions also are universal in the Voronin sense. The attention was also devoted to the periodic zeta-function. The first universality results for periodic zeta-function was obtained by B. Bagchi in [1] and [2], and by different methods in [13] and [14]. We will recall Theorem 11.8 from [14]. Denote by $\mathcal{K}$ the class of compact subsets of the strip $D = \left\{s \in \mathbb{C} : \ \frac{1}{2} < \sigma < 1\right\}$ with connected complements, and by $H(K)$, $K \in \mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K. Let $\operatorname{meas} A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Theorem 1. [14]. *Suppose that a_m is not a multiple of a character* $\bmod k$ *satisfying $a_m = 0$ for $(m,k) > 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \ \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

In [14], also the upper bounds for the density of universality of $\zeta(s;\mathfrak{a})$ were obtained.

We note that the assumptions of Theorem 1 imply that the sequence $\mathfrak{a}$ is not multiplicative. We recall that the sequence $\mathfrak{a}$ is multiplicative if $a_{mn} = a_m a_n$ for all coprime $m, n \in \mathbb{N}$. The universality of $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ was obtained in [11]. Denote by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K, which are analytic in the interior of K.

Theorem 2. [11]. *Suppose that the sequence is multiplicative and*

$$\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\frac{\alpha}{2}}} \leqslant c < 1$$

for all primes p. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the assertion of Theorem 1 is true.

The universality of periodic zeta-functions is not a simple problem. It turns out, as it was observed in [5], that not all periodic zeta-functions are universal in the Voronin sense. Moreover, in [5], a new restricted universality property for $\zeta(s;\mathfrak{a})$ was introduced. For $K \in \mathcal{K}$, the height $h(K)$ of K is defined by

$$h(K) = \max_{s\in K} \operatorname{Im}(s) - \min_{s\in K} \operatorname{Im}(s).$$

Then in [5] the following theorem has been obtained.

Theorem 3. *There exists a positive constant $c = c(\mathfrak{a})$ such that, for every $K \in \mathcal{K}$ of height $h(K) \leqslant c$, every $f(s) \in H(K)$ and every $\varepsilon > 0$, the inequality of Theorem 1 is true.*

Also, in [5], the necessary and sufficient conditions of the universality for $\zeta(s;\mathfrak{a})$ with prime k were obtained. In [15], the universality of the function $\zeta(s;\mathfrak{a})$ with prime k satisfying the condition

$$a_k = \frac{1}{\varphi(k)} \sum_{m=1}^{k-1} a_m,$$

where $\varphi(k)$ is the Euler function, was considered. A joint universality theorem for periodic zeta-functions was proved in [9]. The joint universality of periodic and periodic Hurwitz zeta-functions was studied in [4] and [7].

The aim of this paper is to discuss the weighted universality of the function $\zeta(s;\mathfrak{a})$. The universality of this type for the Riemann zeta-function was considered in [6].

Let $w(t)$ be a positive function of bounded variation on $[T_0,\infty)$, $T_0 > 0$, such that the variation $V_a^b w$ on $[a,b]$ satisfies the inequality $V_a^b w \leqslant cw(a)$ with certain $c > 0$ for any subinterval $[a,b] \subset [T_0,\infty)$. Define $U = U(T,w) = \int_{T_0}^T w(t)\,\mathrm{d}\,t$ and suppose that $U(T,w) \to \infty$ as $T \to \infty$. Let I_A stand for the indicator function of the set A. Then the following statement holds.

Theorem 4. *Suppose that the function $w(t)$ satisfies all above conditions, and that the sequence $\mathfrak{a}$ is as in Theorem 2. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U} \int_{T_0}^T w(\tau) I_{\left\{\tau:\, \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a})-f(s)|<\varepsilon\right\}}(\tau)\,\mathrm{d}\,\tau > 0.$$

We note that in [6] a certain additional condition generalizing the classical Birkhoff-Khintchine theorem was used. We do not need that condition.

2 Limit theorems

Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X, and by $H(D)$ the space of analytic functions on D equipped with the topology of uniform convergence on compacta. This section is devoted to a limit theorem on weakly convergent probability measures in the space $(H(D),\mathcal{B}(H(D)))$.

Let $\gamma \stackrel{def}{=} \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane. Define $\Omega = \prod_p \gamma_p$, where $\gamma_p = \gamma$ for all primes p. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ can be defined. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_p. Moreover, let

$$\omega(m) = \prod_{\substack{p^\alpha | m \\ p^{\alpha+1} \nmid m}} \omega^\alpha(p)$$

for $m \in \mathbb{N}$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \omega; \mathfrak{a})$ by the formula

$$\zeta(s, \omega; \mathfrak{a}) = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{a_{p^\alpha} \omega^\alpha(p)}{p^{\alpha s}} \right).$$

We note that the latter product converges uniformly on compact subsets of D for almost all $\omega \in \Omega$. Moreover, for almost all $\omega \in \Omega$,

$$\zeta(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}.$$

We start with a weighted limit theorem on the torus. Let, for $A \in \mathcal{B}(\Omega)$,

$$Q_{T,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau :\ (p^{-i\tau} :\ p \in \mathcal{P}) \in A\}}(\tau)\, \mathrm{d}\,\tau,$$

where $\mathcal{P}$ is the set of all prime numbers.

Lemma 1. *$Q_{T,w}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. Denote by $g_{T,w}(\underline{k})$, $\underline{k} = (k_p :\ k_p \in \mathbb{Z},\ p \in \mathcal{P})$, the Fourier transform of the measure $Q_{T,w}$. Since characters χ of Ω are of the form

$$\chi(\omega) = \prod_p \omega^{k_p}(p),$$

where only a finite number of integers k_p are distinct from zero, we have that

$$g_{T,w}(\underline{k}) = \int_\Omega \prod_p \omega^{k_p}(p)\, \mathrm{d}\, Q_{T,w}.$$

Hence, by the definition of $Q_{T,w}$,

$$\begin{aligned} g_{T,w}(\underline{k}) &= \frac{1}{U} \int_{T_0}^{T} w(\tau) \prod_p p^{-ik_p\tau}\, \mathrm{d}\,\tau \\ &= \frac{1}{U} \int_{T_0}^{T} w(\tau) \exp\Big\{ -i\tau \sum_p k_p \log p \Big\}\, \mathrm{d}\,\tau, \qquad (2.1) \end{aligned}$$

where only a finite number of integers k_p are distinct from zero. It is well known that the set $\{\log p : \ p \in \mathcal{P}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Therefore, in view of (2.1),

$$g_{T,w}(\underline{0}) = 1 \tag{2.2}$$

and, for $\underline{k} \neq \underline{0}$, using properties of $w(t)$, we find that

$$\begin{aligned} g_{T,w}(\underline{k}) &= -\frac{1}{Ui \sum\limits_{p} k_p \log p} \int_{T_0}^{T} w(\tau)\, \mathrm{d} \exp\Big\{ - i\tau \sum_{p} k_p \log p \Big\} \\ &= O\Big(|U \sum_{p} k_p \log p|^{-1} \Big). \end{aligned}$$

This and (2.2) show that

$$\lim_{T \to \infty} g_{T,w}(\underline{k}) = \begin{cases} 1, & \text{if} \quad \underline{k} = \underline{0}, \\ 0, & \text{if} \quad \underline{k} \neq \underline{0}, \end{cases}$$

i.e., $g_{T,w}(\underline{k})$, as $T \to \infty$, converges to the Fourier transform of the measure m_H. Hence, the lemma follows. □

Let $\theta > \frac{1}{2}$ be a fixed number and, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\Big\{ - \Big(\frac{m}{n}\Big)^{\theta} \Big\}.$$

Define

$$\zeta_n(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^s} \qquad \text{and} \qquad \zeta_n(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m) v_n(m)}{m^s},$$

and let the function $u_n : \ \Omega \to H(D)$ be defined by the formula

$$u_n(\omega) = \zeta_n(s, \omega; \mathfrak{a}).$$

Since the series for $\zeta_n(s, \omega; \mathfrak{a})$ is absolutely convergent for $\sigma > \frac{1}{2}$ [11], the function u_n is continuous one. We set $\hat{P}_n = m_H u_n^{-1}$, where, for $A \in \mathcal{B}(H(D))$,

$$\hat{P}_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1} A).$$

Define

$$P_{T,n,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau :\, \zeta_n(s+i\tau; \mathfrak{a}) \in A\}}(\tau)\, \mathrm{d}\,\tau, \quad A \in \mathcal{B}(H(D)).$$

Lemma 2. *$P_{T,n,w}$ converges weakly to $\hat{P}_n$ as $T \to \infty$.*

Proof. Clearly,

$$u_n\left(p^{-i\tau} : \ p \in \mathcal{P}\right) = \zeta_n(s + i\tau; \mathfrak{a}).$$

Therefore,

$$P_{T,n,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau\colon (p^{-i\tau}\colon p\in\mathcal{P})\in u_n^{-1}A\}}(\tau)\,\mathrm{d}\,\tau$$
$$= Q_{T,w}(u_n^{-1}A) = Q_{T,w}u_n^{-1}(A).$$

This, the continuity of u_n, Lemma 1 and Theorem 5.1 of [3] prove the lemma. □

Now we will approximate $\zeta(s;\mathfrak{a})$ by $\zeta_n(s;\mathfrak{a})$. Let, for $g_1, g_2 \in H(D)$,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l}|g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l}|g_1(s) - g_2(s)|},$$

where $\{K_l :\ l \in \mathbb{N}\}$ is a sequence of compact subsets of the strip D such that $D = \cup_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact, then $K \subset K_l$ for some l. Then ρ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta.

Lemma 3. *The equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau)\rho(\zeta(s+i\tau;\mathfrak{a}), \zeta_n(s+i\tau;\mathfrak{a}))\,\mathrm{d}\,\tau = 0$$

holds.

Proof. Consider the series

$$\sum_{m=1}^{\infty} \frac{b_n(m)}{m^s}, \tag{2.3}$$

where

$$b_n(m) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \frac{a_m l_n(s)}{s m^s}\,\mathrm{d}\,s, \quad l_n(s) = \frac{s}{\theta}\Gamma(s/\theta)\, n^s, \quad n\in\mathbb{N},$$

$\Gamma(s)$ is the Euler gamma-function, and $\theta > \frac{1}{2}$ is as above. Since a_m is uniformly bounded, we find that

$$b_n(m) \ll m^{-\theta}.$$

Thus, the series (2.3) is absolutely convergent for $\sigma > \frac{1}{2}$. From this remark, we deduce that

$$\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z;\mathfrak{a}) l_n(z)\frac{\mathrm{d}\,z}{z} = \sum_{m=1}^{\infty}\frac{b_n(m)}{m^s}, \tag{2.4}$$

and an application of the Mellin formula shows that

$$b_n(m) = a_m \exp\Big\{ -\Big(\frac{m}{n}\Big)^{\theta}\Big\}.$$

Now the series (2.3) coincides with $\zeta_n(s;\mathfrak{a})$. Therefore, by (2.4) and the residue theorem,

$$\zeta_n(s;\mathfrak{a}) = \frac{1}{2\pi i}\int_{\theta-\sigma-i\infty}^{\theta-\sigma+i\infty} \zeta(s+z;\mathfrak{a}) l_n(z)\frac{\mathrm{d}\,z}{z} + \zeta(s;\mathfrak{a}) + \operatorname*{Res}_{z=1-s} \zeta(s+z;\mathfrak{a}) l_n(z)\frac{1}{z}, \tag{2.5}$$

where $\frac{1}{2} < \sigma < 1$ and $\sigma > \theta$.

Suppose that $\sigma \geqslant \frac{1}{2}$ and $2\pi \leqslant |t| \leqslant \pi x$. Then, see, for example, [8],

$$\zeta(s,\alpha) = \sum_{0\leqslant m\leqslant x} \frac{1}{(m+\alpha)^s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}). \tag{2.6}$$

Moreover, by (1.1),

$$\zeta(s;\mathfrak{a}) = O\left(\sum_{m=1}^{k} \left|\zeta\left(s, \tfrac{m}{k}\right)\right|\right).$$

From this and (2.6), we find similarly as in the proof of Lemma 4 of [10] that, for $\frac{1}{2} < \sigma < 1$ and $\tau \in \mathbb{R}$,

$$\int_{T_0+\tau}^{T+\tau} w(t-\tau)|\zeta(\sigma+it;\mathfrak{a})|^2 \,\mathrm{d}\,t = O(U(1+|\tau|)^2).$$

Let K be a compact subset of the strip D. Then, using (2.5) and the contour integration, we obtain that with $\hat{\sigma} < 0$

$$\frac{1}{U}\int_{T_0}^{T} w(\tau)\sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a}) - \zeta_n(s+i\tau;\mathfrak{a})| \,\mathrm{d}\,\tau = O\left(\int_{-\infty}^{\infty} |l_n(\hat{\sigma}+it)|(1+|t|)^2 \,\mathrm{d}\,t\right) + o(1)$$

as $T \to \infty$. This and the definition of $l_n(s)$ prove the lemma. □

Denote by P_ζ the distribution of the random element $\zeta(s,\omega;\mathfrak{a})$, i.e,

$$P_\zeta(A) = m_H(\omega \in \Omega: \ \zeta(s,\omega;\mathfrak{a}) \in A), \quad A \in \mathcal{B}(H(D)).$$

For $A \in \mathcal{B}(H(D))$, define

$$P_{T,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{\tau:\ \zeta(s+i\tau;\mathfrak{a})\in A\}}(\tau)\,\mathrm{d}\,\tau.$$

Theorem 5. *The measure $P_{T,w}$ converges weakly to P_ζ as $T \to \infty$. Moreover, the support of P_ζ is the set $\{g \in H(D): \ g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.*

Proof. On a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$, define a random variable η_T by

$$\mathbb{P}(\eta_T \in A) = \frac{1}{U}\int_{T_0}^{T} w(t) I_A(t)\,\mathrm{d}\,t, \quad A \in \mathcal{B}(\mathbb{R}).$$

By Lemma 2, we have that $P_{T,n,w}$ converges weakly to $\hat{P}_n$ as $T \to \infty$. Define

$$X_{T,n} = X_{T,n}(s) = \zeta_n(s + i\eta_T; \mathfrak{a}).$$

Then the assertion of Lemma 2 can be written as

$$X_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \hat{X}_n, \tag{2.7}$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution, and $\hat{X}_n$ is the $H(D)$-valued random element having the distribution $\hat{P}_n$. We will prove that the family of probability measures $\{\hat{P}_n :\ n \in \mathbb{N}\}$ is tight.

Since the series for $\zeta_n(s; \mathfrak{a})$ is absolutely convergent for $\sigma > \frac{1}{2}$, it is not difficult to see that, for $\sigma > \frac{1}{2}$,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t)|\zeta_n(\sigma + it; \mathfrak{a})|^2 \,\mathrm{d}\, t &= \sum_{m=1}^{\infty} \frac{|a_m|^2 v_n^2(m)}{m^{2\sigma}} \\ &\leqslant \sum_{m=1}^{\infty} \frac{|a_m|^2}{m^{2\sigma}} \leqslant C < \infty. \end{aligned}\tag{2.8}$$

Let K_l be a compact set from the distribution of the metric ρ. Then using the Cauchy integral formula and (2.8) leads to

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) \sup_{s\in K_l} |\zeta_n(s + i\tau; \mathfrak{a})| \,\mathrm{d}\, \tau \leqslant R_l < \infty.$$

Now let $\varepsilon > 0$ be arbitrary and $M_l = 2^l R_l \varepsilon^{-1}$. Then

$$\begin{aligned}&\limsup_{T\to\infty} \mathbb{P}\left(\sup_{s\in K_l} |X_{T,n}(s)| > M_l\right) \\ &\quad = \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\left\{\tau:\ \sup\limits_{s\in K_l} |\zeta_n(s+i\tau;\mathfrak{a})| \geqslant M_l\right\}}(\tau) \,\mathrm{d}\, \tau \\ &\quad \leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{M_l U} \int_{T_0}^{T} w(\tau) \sup_{s\in K_l} |\zeta_n(s + i\tau; \mathfrak{a})| \,\mathrm{d}\, \tau \leqslant \frac{\varepsilon}{2^l}.\end{aligned}$$

Therefore, in view of (2.7),

$$\mathbb{P}\left(\sup_{s\in K_l} |\hat{X}_n(s)| > M_l\right) \leqslant \frac{\varepsilon}{2^l} \tag{2.9}$$

for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$. Let

$$H_\varepsilon = \left\{g \in H(D) :\ \sup_{s\in K_l} |g(s) \leqslant M_l,\ l \in \mathbb{N}\right\}.$$

Then the set H_ε is uniformly bounded on every compact set of D, thus it is compact subset of $H(D)$. Moreover, by (2.9)

$$\mathbb{P}(\hat{X}_n(s) \in H_\varepsilon) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Hence,

$$\hat{P}_n(H_\varepsilon) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$, i.e., the family $\{\hat{P}_n\}$ is tight. Therefore, by the Prokhorov theorem [3], it is relatively compact. Hence, every sequence of $\{\hat{P}_n\}$ contains a subsequence $\{\hat{P}_{n_r}\}$ such that $\hat{P}_{n_r}$ converges weakly to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$, i.e.,

$$\hat{X}_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{2.10}$$

Moreover, using Lemma 3, we find that, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau:\ \rho(\zeta(s+i\tau;\mathfrak{a}),\zeta_n(s+i\tau,\mathfrak{a}))\geqslant\varepsilon\}}(\tau)\,\mathrm{d}\,\tau \\ &\qquad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon U} \int_{T_0}^{T} w(\tau)\rho(\zeta(s+i\tau;\mathfrak{a}),\zeta_n(s+i\tau,\mathfrak{a}))\,\mathrm{d}\,\tau = 0. \end{aligned}$$

Now this, (2.7), (2.10) and Theorem 4.2 of [3] show that

$$X_T(s) = \zeta(s + i\eta_T; \mathfrak{a}) \xrightarrow[T\to\infty]{\mathcal{D}} P.$$

Hence, $P_{T,w}$ converges weakly to P as $T \to \infty$. The latter relation also implies, that the measure P in (2.10) is independent of the choice of subsequence $\hat{P}_{n_r}$. Thus

$$\hat{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P,$$

or $\hat{P}_n$ converges weakly to P. This means that $P_{T,w}$, as $T \to \infty$, converges weakly to the limit measure of $\hat{P}_n$, as $n \to \infty$. It remains to identify the measure P.

In [11], the measure

$$P_T(A) = \frac{1}{T} \operatorname{meas} \{\tau \in [0,T] :\ \zeta(s+i\tau;\mathfrak{a}) \in A\}, \quad A \in \mathcal{B}(H(D))$$

was considered, and it was obtained that P_T converges weakly to P_ζ as $T \to \infty$. Moreover, in the proving process, it was observed that P_T, as $P_{T,w}$, also converges weakly to the limit measure of $\hat{P}_n$ as $n \to \infty$, i.e, to the measure P. From these remarks, we have that P coincides with P_ζ. In [11] it is also noted that the support of the measure P_ζ is the set $\{g \in H(D) :\ g(s) \neq 0 \text{ or } g(s) \equiv 0\}$. The theorem is proved. □

3 Universality

The proof of Theorem 4 is quite standard and is based on Theorem 5 and the Mergelyan theorem on the approximation of analytic functions by polynomials [12].

Proof of Theorem 4. By Theorem 5 and the equivalent of weak convergence of probability measures in terms of open sets [3], we have that

$$\liminf_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{\tau:\ \zeta(s+i\tau;\mathfrak{a})\in G\}}(\tau)\,\mathrm{d}\,\tau \geqslant P_\zeta(G), \tag{3.1}$$

where

$$G = \left\{ g \in H(D):\ \sup_{s\in K} |g(s) - e^{p(s)}| < \frac{\varepsilon}{2} \right\}$$

and $p(s)$ is a polynomial such that

$$\sup_{s\in K} |f(s) - e^{p(s)}| < \frac{\varepsilon}{2}. \tag{3.2}$$

The existence of $p(s)$ is implied by the Mergelyan theorem. By Theorem 5, $e^{p(s)}$ is an element of the support of the measure P_ζ, thus $P_\zeta(G) > 0$ because G is an open neighbourhood of $e^{p(s)}$. Therefore, in view of (3.1) and the definition of G,

$$\liminf_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) I_{\left\{\tau:\ \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - e^{p(s)}| < \frac{\varepsilon}{2}\right\}}(\tau)\,\mathrm{d}\,\tau > 0.$$

From this, the theorem follows since, in virtue of (3.2),

$$\left\{ \tau:\ \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - e^{p(s)}| < \frac{\varepsilon}{2} \right\} \subset \left\{ \tau:\ \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon \right\}.$$

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