A Study of Nonlinear Fractional-Order Boundary Value Problem with Nonlocal Erdélyi-Kober and Generalized Riemann-Liouville Type Integral Boundary Conditions

Bashir Ahmad[a], Sotiris K. Ntouyas[b,a], Jessada Tariboon[c] and Ahmed Alsaedi[a]

[a] *Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group*
  Department of Mathematics, Faculty of Science, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[b] *University of Ioannina*
  Department of Mathematics, 451 10 Ioannina, Greece
[c] *Nonlinear Dynamic Analysis Research Center*
  Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800 Thailand
  E-mail(*corresp.*): `bashirahmad_qau@yahoo.com`
  E-mail: `aalsaedi@hotmail.com`
  E-mail: `sntouyas@uoi.gr`
  E-mail: `jessada.t@sci.kmutnb.ac.th`

Abstract. We investigate a new kind of nonlocal boundary value problems of nonlinear Caputo fractional differential equations supplemented with integral boundary conditions involving Erdélyi-Kober and generalized Riemann-Liouville fractional integrals. Existence and uniqueness results for the given problem are obtained by means of standard fixed point theorems. Examples illustrating the main results are also discussed.

Keywords: Caputo fractional derivative, Erdélyi-Kober, generalized Riemann-Liouville fractional integral, fractional integral, fixed point.

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1 Introduction

Hemodynamic conditions (blood velocity and pressure as a function of space and time) can be completely characterized by computational fluid dynamics (CFD) techniques under appropriate boundary data. The idea of assuming the

geometry of blood vessel to be of circular type upstream the inlet is not always justifiable. An alternative and accurate approach to model the blood flow problems is to consider integral boundary conditions, for details, see [2] and references cited therein. Integral boundary conditions also help to regularize ill-posed parabolic backward problems in time partial differential equations [33].

The recent surge in developing theoretical and numerical aspects of fractional calculus is mainly due to widespread applications of fractional-order operators in mathematical modeling of several real world phenomena. Examples include acoustics, viscoelasticity, signal and image processing, chemical processes, control theory, biomathematics, biomedical, etc. For more details and explanations, see the texts [16,18,23,26,27]. We emphasize that fractional-order differential and integral operators help to gather useful information about hereditary and memory characteristics of the processes and materials involved in the phenomena.

Fractional differential equations supplemented with a variety of initial and boundary conditions have been investigated by several researchers and the literature on the topic is now much enriched. For examples and recent development of the topic, see [1,3,4,5,6,7,9,12,13,21,22,24,25,30,31,32] and the references cited therein.

Classical, Riemann-Liouville or Hadamard or Erdélyi-Kober type integrals appear in the study of fractional-order boundary value problems with integral boundary conditions. In [15], the author expressed Riemann-Liouville and Hadamard fractional integrals into a single form, which is known as generalized Riemann-Liouville fractional integral (see Definition 2).

In this paper, we introduce a new class of boundary value problems of Caputo fractional differential equations supplemented with Erdélyi-Kober and generalized Riemann-Liouville fractional integral boundary conditions. In precise terms, we consider the following boundary value problem

$$
\begin{cases}
{}^{c}D^{q}x(t) = f(t, x(t)), \quad t \in [0, T], \\[2mm]
x(0) = \alpha \dfrac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \displaystyle\int_{0}^{\zeta} \dfrac{s^{\eta\gamma+\eta-1}x(s)}{(\zeta^{\eta}-s^{\eta})^{1-\delta}} ds := \alpha I_{\eta}^{\gamma,\delta}x(\zeta), \\[4mm]
x(T) = \beta \dfrac{\rho^{1-p}}{\Gamma(p)} \displaystyle\int_{0}^{\xi} \dfrac{s^{\rho-1}x(s)}{(\xi^{\rho}-s^{\rho})^{1-p}} ds := \beta \, {}^{\rho}I^{p}x(\xi), \quad 0 < \zeta, \xi < T,
\end{cases}
\tag{1.1}
$$

where ${}^{c}D^{q}$ is the Caputo fractional derivative of order $1 < q \le 2$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_{\eta}^{\gamma,\delta}$ is Erdélyi-Kober fractional integrals of order $\delta > 0$, $\eta > 0$, $\gamma \in \mathbb{R}$, ${}^{\rho}I^{p}$ is the generalized Riemann-Liouville fractional integral of order $p > 0$, $\rho > 0$ and $\alpha, \beta \in \mathbb{R}$.

We remark that Erdélyi-Kober fractional integral operator, introduced by Arthur Erdélyi and Hermann Kober [11] in 1940, is useful in solving single, dual and triple integral equations possessing special functions of mathematical physics in their kernels. For details and applications of the Erdélyi-Kober fractional integrals, for instance, see [10,17,19,29,34].

The paper is organized as follows: In Section 2 we will present some useful preliminaries and lemmas. Section 3 deals with the existence and uniqueness

results for the boundary value problem (1.1) which are established via Krasnoselskii fixed point theorem, Laray-Schauder nonlinear alternative, contraction mapping principle, nonlinear contractions, Schaefer fixed point theorem, and Laray-Schauder degree theory. The paper concludes with illustrative examples.

2 Preliminaries

This section is devoted to some preliminary concepts of fractional calculus that we need in the forthcoming analysis [16].

DEFINITION 1. The fractional integral of order q with the lower limit zero for a function f is defined as

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > 0, \quad q > 0,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(q) = \int_0^\infty t^{q-1} e^{-t} dt$.

DEFINITION 2. [15] The generalized fractional integral of order $q > 0$ and $\rho > 0$ of a function $f(t)$, for all $0 < t < \infty$, is defined as

$$^\rho I^q f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{1-q}} ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Remark 1. From the above definition it follows that $\rho = 1$ corresponds to the standard Riemann-Liouville fractional integral, while $\rho \to 0$ yields the Hadamard fractional integral

$$\lim_{\rho \to 0} {}^\rho I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\ln \frac{t}{s} \right)^{q-1} \frac{f(s)}{s} ds.$$

For further details, see [15].

DEFINITION 3. The Erdélyi-Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I_\eta^{\gamma,\delta} f(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} f(s)}{(t^\eta - s^\eta)^{1-\delta}} ds,$$

provided the right side is pointwise defined on $\mathbb{R}_+$.

Remark 2. For $\eta = 1$ the above operator is reduced to the Kober operator

$$I_1^{\gamma,\delta} f(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^\gamma f(s)}{(t-s)^{1-\delta}} ds, \quad \gamma, \ \delta > 0,$$

that was introduced for the first time by Kober in [19]. For $\gamma = 0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight:

$$I_1^{0,\delta} f(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{f(s)}{(t-s)^{1-\delta}} ds, \quad \delta > 0.$$

DEFINITION 4. The Riemann-Liouville fractional derivative of order $q > 0$, $n-1 < q < n$, $n \in N$, is defined as

$$D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

DEFINITION 5. The Caputo derivative of order q for a function $f : [0,\infty) \to \mathbb{R}$ can be written as

$${}^c D^q f(t) = D_{0+}^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n-1 < q < n.$$

Remark 3. If $f(t) \in C^n[0,\infty)$, then

$${}^c D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds = I^{n-q} f^{(n)}(t), \ t > 0, \ n-1 < q < n.$$

Lemma 1. *Let $\delta, \eta > 0$ and $\gamma, q \in \mathbb{R}$. Then we have*

$$I_\eta^{\gamma,\delta} t^q = \frac{t^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(\gamma + (q/\eta) + \delta + 1)}.$$

Proof. Recall the definition of beta function and its property:

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} du \quad \text{and} \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

for $x, y > 0$. From Definition 3, we have

$$
\begin{aligned}
I_\eta^{\gamma,\delta} t^q &= \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} \cdot s^q}{(t^\eta - s^\eta)^{1-\delta}} ds = \frac{t^q}{\Gamma(\delta)} \int_0^1 u^{\gamma+\frac{q}{\eta}}(1-u)^{\delta-1} du \\
&= \frac{t^q}{\Gamma(\delta)} B\left(\gamma + \frac{q}{\eta} + 1, \delta\right) = \frac{t^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(\gamma + (q/\eta) + \delta + 1)}.
\end{aligned}
$$

The proof is complete. $\square$

Lemma 2. *Let constants $q > 0$ and $p > 0$. Then:*

$${}^\rho I^q t^p = \frac{\Gamma(p + \rho/\rho)}{\Gamma((p + \rho q + \rho)/\rho)} \frac{t^{p+\rho q}}{\rho^q}.$$

Proof. From Definition 2, we have

$$
\begin{aligned}
{}^\rho I^q t^p &= \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} s^p}{(t^\rho - s^\rho)^{1-q}} ds = \frac{\rho^{1-q}}{\Gamma(q)} \frac{t^{p+\rho q}}{\rho} \int_0^1 \frac{u^{\frac{p}{\rho}}}{(1-u)^{1-q}} du \\
&= \frac{\rho^{1-q}}{\Gamma(q)} \frac{t^{p+\rho q}}{\rho} B\left(\frac{p+\rho}{\rho}, q\right) = \frac{t^{p+\rho q}}{\rho^q} \frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)}.
\end{aligned}
$$

This completes the proof. $\square$

Lemma 3. *For any $y \in C[0,T]$, a function $x \in C^2[0,T]$ is a solution of the linear fractional boundary value problem*

$$
\begin{cases}
{}^cD^q x(t) = y(t), & 1 < q \le 2, \\
x(0) = \alpha I_\eta^{\gamma,\delta} x(\zeta), & x(T) = \beta\ {}^\rho I^p x(\xi), \quad 0 < \zeta, \xi < T,
\end{cases}
\tag{2.1}
$$

if and only if

$$
\begin{aligned}
x(t) = J^q y(t) &+ \frac{\alpha}{\Lambda}(v_4 - tv_3) I_\eta^{\gamma,\delta} J^q y(\zeta) \\
&+ \frac{1}{\Lambda}(v_2 + tv_1)\Big(\beta\ {}^\rho I^p J^q y(\xi) - J^q y(T)\Big),
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
v_1 &= 1 - \alpha\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)}, & v_2 &= \alpha\zeta\frac{\Gamma(\gamma+1/\eta+1)}{\Gamma(\gamma+1/\eta+\delta+1)}, \\
v_3 &= 1 - \beta\frac{\xi^{\rho p}}{\rho^\beta}\frac{1}{\Gamma(p+1)}, & v_4 &= T - \beta\frac{\xi^{\rho p+1}}{\rho^p}\frac{\Gamma\big((1+\rho)/\rho\big)}{\Gamma\big((1+\rho p+\rho)/\rho\big)}
\end{aligned}
\tag{2.3}
$$

and $\Lambda = v_1 v_4 + v_2 v_3 \neq 0$.

Proof. It is well known [16] that the general solution of fractional differential equation in (2.1) can be written as

$$
x(t) = c_0 + c_1 t + J^q y(t),
\tag{2.4}
$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants.

Applying the Erdélyi-Kober fractional integral operator of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$ on (2.4) and using Lemma 1, we obtain

$$
I_\eta^{\gamma,\delta} x(t) = I_\eta^{\gamma,\delta} J^q y(t) + c_1 t\frac{\Gamma(\gamma+1/\eta+1)}{\Gamma(\gamma+1/\eta+\delta+1)} + c_0\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)}.
\tag{2.5}
$$

Using (2.5) in the first condition of (2.1), we get

$$
c_0 = \alpha I_\eta^{\gamma,\delta} J^q y(\zeta) + \alpha c_0\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)} + \alpha c_1 \zeta\frac{\Gamma(\gamma+1/\eta+1)}{\Gamma(\gamma+1/\eta+\delta+1)}.
\tag{2.6}
$$

Applying the generalized fractional integral operator on (2.4) and using Lemma 2, we obtain

$$
{}^\rho I^p x(t) = {}^\rho I^p J^q y(t) + c_0\frac{t^{\rho p}}{\rho^p}\frac{1}{\Gamma(p+1)} + c_1\frac{t^{\rho p+1}}{\rho^p}\frac{\Gamma((1+\rho)/\rho)}{\Gamma((1+\rho p+\rho)/\rho)}.
\tag{2.7}
$$

Using (2.7) in the second boundary condition of (2.1), we get

$$
J^q y(T) + c_0 + c_1 T = \beta\ {}^\rho I^p J^q y(\xi) + c_0\beta\frac{\xi^{\rho p}}{\rho^p}\frac{1}{\Gamma(p+1)} + c_1\beta\frac{\xi^{\rho p+1}}{\rho^p}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho p+\rho}{\rho})}.
\tag{2.8}
$$

Using the notations (2.3) in (2.6) and (2.8), we get the system

$$v_1 c_0 - v_2 c_1 = \alpha I_\eta^{\gamma,\delta} J^q y(\zeta),$$
$$v_3 c_0 + v_4 c_1 = \beta \,{}^\rho I^p J^q y(\xi) - J^q y(T),$$

which, on solving for c_0 and c_1, yields

$$c_0 = \frac{1}{\Lambda}\Big\{\alpha v_4 I_\eta^{\gamma,\delta} J^q y(\zeta) + v_2\Big(\beta \,{}^\rho I^p J^q y(\xi) - J^q y(T)\Big)\Big\},$$

$$c_1 = \frac{1}{\Lambda}\Big\{v_1\Big(\beta \,{}^\rho I^p J^q y(\xi) - J^q y(T)\Big) - \alpha v_3 I_\eta^{\gamma,\delta} J^q y(\zeta)\Big\}.$$

Substituting the values of c_0, c_1 in (2.4), we get (2.2). Conversely, it follows by direct computation that the integral equation (2.2) satisfies the problem (2.1). This completes the proof. □

3   Existence results

We denote by $\mathcal{C} = C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$. Also by $L^1([0,T],\mathbb{R})$ we denote the Banach space of measurable functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)|dt$.

In view of Lemma 3, we define an operator $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ by

$$(\mathcal{P}x)(t) = J^q f(s, x(s))(t) + \frac{\alpha}{\Lambda}(v_4 - tv_3) I_\eta^{\gamma,\delta} J^q f(s, x(s))(\zeta)$$

$$+ \frac{1}{\Lambda}(v_2 + tv_1)\Big(\beta \,{}^\rho I^p J^q f(s, x(s))(\xi) - J^q f(s, x(s))(T)\Big), \ t \in [0,T]. \quad (3.1)$$

In the sequel, we use the following expressions:

$$^\rho I^p f(s, x(s))(y) = \frac{\rho^{1-p}}{\Gamma(p)} \int_0^y \frac{s^{\rho-1} f(s, x(s))}{(y^\rho - s^\rho)^{1-p}} ds,$$

for $y \in [0,T]$ and

$$I_\eta^{\gamma,\delta} J^q f(s, x(s))(\xi) = \frac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(q)\Gamma(\delta)} \int_0^\xi \int_0^r \frac{r^{\eta\gamma+\eta-1}(r-s)^{q-1}}{(\xi^\eta - r^\eta)^{1-\delta}} f(s, x(s)) ds\, dr,$$

where $\xi \in (0,T)$.

For convenience, we set a constant

$$\Psi : \ = \frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|v_4| + T|v_3|)}{|\Lambda|} \frac{\zeta^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)}$$

$$+ \frac{(|v_2| + T|v_1|)}{|\Lambda|}\left(\frac{1}{\Gamma(q+1)} \frac{\xi^{q+\rho p}}{\rho^p} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho p+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)}\right). \quad (3.2)$$

In the following subsections we present our main results for problem (1.1) by making use of a variety of fixed point theorems.

3.1 Existence result via Krasnoselskii's fixed point theorem

Lemma 4. *(Krasnoselskii's fixed point theorem) [20]. Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax + By \in M$ whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 1. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumptions:*

(H_1) there exists a positive constant L such that $|f(t, x) - f(t, y)| \leq L|x - y|$, for each $t \in [0, T]$ and $x, y \in \mathbb{R}$.

(H_2) $|f(t, x)| \leq \varphi(t), \quad \forall(t, x) \in [0, T] \times \mathbb{R}$ and $\varphi \in C([0, T], \mathbb{R}^+)$.

Then problem (1.1) has at least one solution on $[0, T]$ provided that

$$
\Psi_1 := L \left\{ \frac{|\alpha|(|v_4| + T|v_3|)}{|\Lambda|} \frac{\zeta^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)} \right. \tag{3.3}
$$
$$
\left. + \frac{(|v_2| + T|v_1|)}{|\Lambda|} \left(\frac{1}{\Gamma(q+1)} \frac{\xi^{q+\rho p}}{\rho^p} \frac{\Gamma((q+\rho)/\rho)}{\Gamma((q+\rho p + \rho)/\rho)} + \frac{T^q}{\Gamma(q+1)} \right) \right\} < 1.
$$

Proof. We define operator $\mathcal{P}_1, \mathcal{P}_2 : \mathcal{C} \to \mathcal{C}$ by

$$
\mathcal{P}_1 x(t) = J^q f(s, x(s))(t), \quad t \in [0, T],
$$
$$
\mathcal{P}_2 y(t) = \frac{\alpha}{\Lambda}(v_4 - tv_3) I_\eta^{\gamma, \delta} J^q f(s, y(s))(\zeta)
$$
$$
+ \frac{1}{\Lambda}(v_2 + tv_1)\left(\beta \, {}^\rho I^p J^q f(s, y(s))(\xi) - J^q f(s, y(s))(T)\right), \quad t \in [0, T].
$$

Setting $\sup_{t \in [0,T]} \varphi(t) = \|\varphi\|$ and choosing $\rho \geq \|\varphi\|\Psi$, where Ψ is defined by (3.2), we consider $B_\rho = \{x \in \mathcal{C} : \|x\| \leq \rho\}$. For any $x, y \in B_\rho$, we have

$$
|\mathcal{P}_1 x(t) + \mathcal{P}_2 y(t)| \leq \sup_{t \in [0,T]} \left\{ J^q |f(s, x(s))|(t) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|) \right.
$$
$$
\times I_\eta^{\gamma, \delta} J^q |f(s, y(s))|(\zeta) + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)
$$
$$
\left. \times \left(|\beta| \, {}^\rho I^p J^q |f(s, y(s))|(\xi) + J^q |f(s, y(s))|(T)\right) \right\}
$$
$$
\leq \|\varphi\| \left\{ \frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|v_4| + T|v_3|)}{|\Lambda|} \frac{\zeta^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)} \right.
$$
$$
\left. + \frac{(|v_2| + T|v_1|)}{|\Lambda|} \left(\frac{1}{\Gamma(q+1)} \frac{\xi^{q+\rho p}}{\rho^p} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho p + \rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)} \right) \right\} = \|\varphi\|\Psi \leq \rho.
$$

This shows that $\mathcal{P}_1 x + \mathcal{P}_2 y \in B_\rho$. Using the assumption ($H_1$), we get

$$
\|\mathcal{P}_2 x - \mathcal{P}_2 y\| \leq L \left\{ \frac{|\alpha|(|v_4| + T|v_3|)}{|\Lambda|} \frac{\zeta^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)} \right.
$$

$$+ \frac{(|v_2| + T|v_1|)}{|\Lambda|} \left(\frac{1}{\Gamma(q+1)} \frac{\xi^{q+\rho p}}{\rho^p} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho p+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)} \right) \bigg\} \|x - y\|,$$

which, in view of (3.3), implies that $\mathcal{P}_2$ is a contraction.

Continuity of f implies that the operator $\mathcal{P}_1$ is continuous. Also, $\mathcal{P}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{P}_1 x\| \le \frac{T^q}{\Gamma(q+1)} \|\varphi\|.$$

Next we prove the compactness of the operator $\mathcal{P}_1$. Let $\sup_{(t,x)\in[0,T]\times B_\rho} |f(t,x)| = \bar{f} < \infty$. Then, for $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we obtain

$$|\mathcal{P}_1 x(t_2) - \mathcal{P}_1 x(t_1)| = \left| J^q f(s, x(s))(t_2) - J^q f(s, x(s))(t_1) \right|$$

$$\le \frac{\bar{f}}{\Gamma(q)} \left| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \right|$$

$$\le \frac{\bar{f}}{\Gamma(q+1)} [|t_2^q - t_1^q| + 2|t_2 - t_1|^q],$$

which tends to zero as $t_2 - t_1 \to 0$ independent of x. Thus, $\mathcal{P}_1$ is equicontinuous. So $\mathcal{P}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{P}_1$ is compact on B_ρ. Thus all the assumptions of Lemma 4 are satisfied. Hence the conclusion of Lemma 4 implies that problem (1.1) has at least one solution on $[0, T]$. □

3.2   Existence result via Leray-Schauder's Nonlinear Alternative

Lemma 5. *(Nonlinear alternative for single valued maps) [14]. Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $\mathcal{A} : \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{A}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) *$\mathcal{A}$ has a fixed point in $\bar{U}$, or*

(ii) *there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $x = \lambda \mathcal{A}(x)$.*

Theorem 2. *Assume that*

(H_3) *there exists a continuous nondecreasing function $\Phi : [0, \infty) \to (0, \infty)$ and a function $p \in L^1([0, T], \mathbb{R}^+)$ such that*

$$|f(t, x)| \le p(t)\Phi(\|x\|) \quad \text{for each} \quad (t, x) \in [0, T] \times \mathbb{R};$$

(H_4) *there exists a constant $N > 0$ such that*

$$N/\Phi(N) > J^q p(s)(T) + A_1 + A_2,$$

where

$$A_1 = |\alpha|/|\Lambda|(|v_4| + T|v_3|)I_\eta^{\gamma,\delta} J^q p(s)(\zeta),$$
$$A_2 = 1/|\Lambda|(|v_2| + T|v_1|)\big(|\beta| \, {}^\rho I^p J^q p(s)(\xi) + J^q p(s)(T)\big).$$

Then the boundary value problem (1.1) has at least one solution on $[0, T]$.

Proof. We first show that the operator $\mathcal{P}$ (defined by (3.1)) maps bounded sets (balls) into bounded sets in $\mathcal{C}$. For a positive constant r, let $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ be a bounded ball in $\mathcal{C}$. Then for $t \in [0, T]$, we have

$$
\begin{aligned}
|\mathcal{P}x(t)| &\le J^q|f(s, x(s))|(t) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)I_\eta^{\gamma,\delta}J^q|f(s, x(s))|(\zeta) \\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta|\,{}^\rho I^p J^q|f(s, x(s))|(\xi) + J^q|f(s, x(s))|(T)\Big) \\
&\le \Phi(\|x\|)J^q p(s)(T) + \Phi(\|x\|)\frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)I_\eta^{\gamma,\delta}J^q p(s)(\zeta) \\
&\quad + \Phi(\|x\|)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta|\,{}^\rho I^p J^q p(s)(\xi) + J^q p(s)(T)\Big)
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
\|\mathcal{P}x\| &\le \Phi(r)\Big\{J^q p(s)(T) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)I_\eta^{\gamma,\delta}J^q p(s)(\zeta) \\
&\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta|\,{}^\rho I^p J^q p(s)(\xi) + J^q p(s)(T)\Big)\Big\}.
\end{aligned}
$$

Next we will show that the operator $\mathcal{P}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$
\begin{aligned}
|\mathcal{P}x(\tau_2) - \mathcal{P}x(\tau_1)| &\le |J^q f(s, x(s))(\tau_2) - J^q f(s, x(s))(\tau_1)| \\
&\quad + \frac{|\alpha||v_3||\tau_2 - \tau_1|}{|\Lambda|}I_\eta^{\gamma,\delta}J^q|f(s, x(s))|(T) \\
&\quad + \frac{|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\beta|\,{}^\rho I^p J^q|f(s, x(s))|(\xi) + J^q|f(s, x(s))|(T)\Big) \\
&\le \frac{\Phi(r)}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]p(s)ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}p(s)ds\right| \\
&\quad + \frac{\Phi(r)|\alpha||v_3||\tau_2 - \tau_1|}{|\Lambda|}I_\eta^{\gamma,\delta}J^q p(s)(T) \\
&\quad + \frac{\Phi(r)|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\beta|\,{}^\rho I^p J^q p(s)(\xi) + J^q p(s)(T)\Big).
\end{aligned}
$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore, by the Arzelá-Ascoli theorem, the operator $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Finally, we show that there exists an open set $U \subset \mathcal{C}$ with $x \ne \theta \mathcal{P}x$ for $\theta \in (0, 1)$ and $x \in \partial U$. Let x be a solution. Then, for $t \in [0, T]$, and following the similar computations as in the first step, we have

$$
|x(t)| \le \Phi(\|x\|)\Big\{J^q p(s)(T) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)I_\eta^{\gamma,\delta}J^q p(s)(\zeta)
$$

$$+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\beta|\ {}^{\rho}I^{p}J^{q}p(s)(\xi)+J^{q}p(s)(T)\Big)\Big\},$$

which leads to

$$\|x\|/\Phi(\|x\|)\leq J^{q}p(s)(T)+A_1+A_2.$$

In view of (H_4), there exists N such that $\|x\|\neq N$. Let us set

$$U=\{x\in\mathcal{C}:\|x\|<N\}.$$

We see that the operator $\mathcal{P}:\overline{U}\rightarrow C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x=\theta\mathcal{P}x$ for some $\theta\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{P}$ has a fixed point $x\in\overline{U}$ which is a solution of the boundary value problem (1.1). This completes the proof. □

3.3 Existence and uniqueness result via Banach's fixed point theorem

Theorem 3. *Let $f:[0,T]\times\mathbb{R}\rightarrow\mathbb{R}$ be a continuous function satisfying the assumptions (H_1). Then there exists a unique solution for problem (1.1) on $[0,T]$ provided that $L\Psi<1$, where Ψ is defined by (3.2).*

Proof. Using the operator $\mathcal{P}$ defined by (3.1), we transform problem (1.1) into a fixed point problem as $x=\mathcal{P}x$. Observe that the fixed points of the operator $\mathcal{P}$ are solutions of problem (1.1). Applying the Banach contraction mapping principle, we shall show that $\mathcal{P}$ has a unique fixed point.

Letting $\sup_{t\in[0,T]}|f(t,0)|=M<\infty$ and $r\geq M\Psi/(1-L\Psi)$, we define a closed ball $B_r=\{x\in\mathcal{C}:\|x\|\leq r\}$ and show that $\mathcal{P}B_r\subset B_r$. Then, using $|f(s,x(s))|\leq|f(s,x(s))-f(s,0)|+|f(s,0)|\leq L\|x\|+M\leq Lr+M$, for any $x\in B_r$, we have

$$|(\mathcal{P}x)(t)|\leq\sup_{t\in[0,T]}\Big\{J^{q}|f(s,x(s))|(t)+\frac{|\alpha|}{|\Lambda|}(|v_4|+T|v_3|)I_{\eta}^{\gamma,\delta}J^{q}|f(s,x(s))|(\zeta)$$

$$+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\beta|\ {}^{\rho}I^{p}J^{q}|f(s,x(s))|(\xi)+J^{q}|f(s,x(s))|(T)\Big)\Big\}$$

$$\leq(L\|x\|+M)J^{q}(1)(T)+(L\|x\|+M)\frac{|\alpha|}{|\Lambda|}(|v_4|+T|v_3|)I_{\eta}^{\gamma,\delta}J^{q}(1)(\zeta)$$

$$+(L\|x\|+M)\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\beta|\ {}^{\rho}I^{p}J^{q}(1)(\xi)+J^{q}(1)(T)\Big)$$

$$\leq(Lr+M)\Big\{\frac{T^{q}}{\Gamma(q+1)}+\frac{|\alpha|(|v_4|+T|v_3|)}{|\Lambda|}\frac{\zeta^{q}\Gamma(\gamma+(q/\eta)+1)}{\Gamma(q+1)\Gamma(\gamma+(q/\eta)+\delta+1)}$$

$$+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\Big(\frac{1}{\Gamma(q+1)}\frac{\xi^{q+\rho p}}{\rho^{p}}\frac{\Gamma(q+\rho/\rho)}{\Gamma(q+\rho p+\rho/\rho)}+\frac{T^{q}}{\Gamma(q+1)}\Big)\Big\}$$

$$\leq(Lr+M)\Psi\leq r,$$

which implies that $\mathcal{P}B_r\subset B_r$.

Next, we let $x, y \in \mathcal{C}$. Then for $t \in [0, T]$, we have

$$
\begin{aligned}
|\mathcal{P}x(t) - \mathcal{P}y(t)| &\leq \sup_{t \in [0,T]} \Big\{ J^q |f(s, x(s)) - f(s, y(s))|(t) \\
&\quad + \frac{|\alpha|}{|\Lambda|} (|v_4| + T|v_3|) I_\eta^{\gamma, \delta} J^q |f(s, x(s)) - f(s, y(s))|(\zeta) \\
&\quad + \frac{1}{|\Lambda|} (|v_2| + T|v_1|) \Big(\beta \, {}^\rho I^p J^q |f(s, x(s)) - f(s, y(s))|(\xi) \\
&\quad + J^q |f(s, x(s)) - f(s, y(s))|(T) \Big) \Big\} \\
&\leq L\|x - y\| J^q(1)(T) + L\|x - y\| \frac{|\alpha|}{|\Lambda|} (|v_4| + T|v_3|) I_\eta^{\gamma, \delta} J^q(1)(\zeta) \\
&\quad + L\|x - y\| \frac{1}{|\Lambda|} (|v_2| + T|v_1|) \Big(|\beta| \, {}^\rho I^p J^q(1)(\xi) + J^q(1)(T) \Big) \\
&= L\Psi \|x - y\|,
\end{aligned}
$$

which leads to $\|\mathcal{P}x - \mathcal{P}y\| \leq L\Psi \|x - y\|$. As $L\Psi < 1$, $\mathcal{P}$ is a contraction. Therefore, we deduce by Banach's contraction mapping principle that $\mathcal{P}$ has a fixed point which corresponds to the unique solution of problem (1.1). The proof is completed. $\square$

3.4 Existence and uniqueness result via nonlinear contractions

DEFINITION 6. Let E be a Banach space and let $\mathcal{F} : E \to E$ be a mapping. $\mathcal{F}$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:

$$
\|\mathcal{F}x - \mathcal{F}y\| \leq \Theta(\|x - y\|), \qquad \forall x, y \in E.
$$

Lemma 6. *(Boyd and Wong) [8]. Let E be a Banach space and let $\mathcal{F} : E \to E$ be a nonlinear contraction. Then $\mathcal{F}$ has a unique fixed point in E.*

Theorem 4. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_5) $|f(t, x) - f(t, y)| \leq z(t) \dfrac{|x - y|}{A^* + |x - y|}$, *for $t \in [0, T]$, $x, y \geq 0$,*
   where $z : [0, T] \to \mathbb{R}^+$ is continuous and

$$
\begin{aligned}
A^* : &= J^q z(T) + \frac{1}{|\Lambda|} \Big[|\alpha|(|v_4| + T|v_3|) I_\eta^{\gamma, \delta} J^q z(\zeta) \\
&\quad + (|v_2| + T|v_1|) \big(|\beta| \, {}^\rho I^p J^q z(\xi) + J^q z(T) \big) \Big].
\end{aligned}
$$

Then problem (1.1) has a unique solution on $[0, T]$.

Proof. Let us introduce a continuous nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ as follows

$$
\Theta(\varepsilon) = A^* \varepsilon / (A^* + \varepsilon), \quad \forall \varepsilon \geq 0.
$$

Note that the function Θ satisfies $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$.

For any $x, y \in \mathcal{C}$ and for each $t \in [0, T]$, we have

$$
\begin{aligned}
|\mathcal{P}x(t) - \mathcal{P}y(t)| \leq {} & \sup_{t \in [0,T]} \Big\{ J^q |f(s, x(s)) - f(s, y(s))|(t) \\
& + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|) I_\eta^{\gamma, \delta} J^q |f(s, x(s)) - f(s, y(s))|(\zeta) \\
& + \frac{1}{|\Lambda|}(|v_2| + T|v_1|) \Big(|\beta| \, {}^\rho I^p J^q |f(s, x(s)) - f(s, y(s))|(\xi) \\
& + J^q |f(s, x(s)) - f(s, y(s))|(T) \Big) \Big\} \\
\leq {} & J^q \left(z(s) \frac{|x - y|}{A^* + |x - y|} \right)(T) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|) I_\eta^{\gamma, \delta} \\
& \times J^q \left(z(s) \frac{|x - y|}{A^* + |x - y|} \right)(\zeta) + \frac{1}{|\Lambda|}(|v_2| + T|v_1| \\
& \times \Big\{ |\beta| \, {}^\rho I^p J^q \left(z(s) \frac{|x - y|}{A^* + |x - y|} \right)(\xi) + J^q \left(z(s) \frac{|x - y|}{A^* + |x - y|} \right)(T) \Big\} \\
\leq {} & \frac{\Theta(\|x - y\|)}{A^*} \Big[J^q z(T) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|) I_\eta^{\gamma, \delta} J^q z(\zeta) \\
& + \frac{1}{|\Lambda|}(|v_2| + T|v_1|) \Big\{ |\beta| \, {}^\rho I^p J^q z(\xi) + J^q z(T) \Big\} \Big] = \Theta(\|x - y\|).
\end{aligned}
$$

This implies that $\|\mathcal{P}x - \mathcal{P}y\| \leq \Theta(\|x - y\|)$. Therefore $\mathcal{P}$ is a nonlinear contraction. Hence, by Lemma 6 the operator $\mathcal{P}$ has a unique fixed point which is the unique solution of problem (1.1). This completes the proof. $\square$

3.5   Existence result via Schaefer fixed point theorem

Lemma 7. *[28] Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X \mid u = \mu Tu, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in X.*

Theorem 5. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exists a positive constant L_1 such that $|f(t, x)| \leq L_1$ for $t \in [0, T]$, $x \in \mathbb{R}$. Then the boundary value problem (1.1) has at least one solution on $[0, T]$.*

Proof. In the first step, we show that the operator $\mathcal{P}$ defined by (3.1) is completely continuous. Observe that continuity of f implies continuity of $\mathcal{P}$. For a positive constant r, let $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ be a bounded set in $\mathcal{C}$. Then, for $t \in [0, T]$, we obtain

$$
\begin{aligned}
|\mathcal{P}x(t)| \leq {} & J^q |f(s, x(s))|(t) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|) I_\eta^{\gamma, \delta} J^q |f(s, x(s))|(\zeta) \\
& + \frac{1}{|\Lambda|}(|v_2| + T|v_1|) \Big(|\beta| \, {}^\rho I^p J^q |f(s, x(s))|(\xi) + J^q |f(s, x(s))|(T) \Big) \\
\leq {} & L_1 J^q(1)(T) + L_1 \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|) I_\eta^{\gamma, \delta} J^q(1)(\zeta)
\end{aligned}
$$

$$+ L_1 \frac{1}{|\varLambda|}(|v_2| + T|v_1|)\Big(|\beta| \; {}^\rho I^p J^q(1)(\xi) + J^q(1)(T)\Big)$$

$$\leq L_1 \Bigg\{ \frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|v_4| + T|v_3|)}{|\varLambda|} \frac{\zeta^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)}$$

$$+ \frac{(|v_2| + T|v_1|)}{|\varLambda|} \Bigg(\frac{1}{\Gamma(q+1)} \frac{\xi^{q+\rho p}}{\rho^p} \frac{\Gamma((q+\rho)/\rho)}{\Gamma((q+\rho p + \rho\rho)/)} + \frac{T^q}{\Gamma(q+1)} \Bigg) \Bigg\},$$

which, by virtue of (3.2), yields

$$\|\mathcal{P}x\| \leq L_1 \varPsi.$$

Next we show that *the operator $\mathcal{P}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$.* Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$|\mathcal{P}x(\tau_2) - \mathcal{P}x(\tau_1)| \leq |J^q f(s, x(s))(\tau_2) - J^q f(s, x(s))(\tau_1)|$$

$$+ \frac{|\alpha||v_2||\tau_2 - \tau_1|}{|\varLambda|} I_\eta^{\gamma,\delta} J^q |f(s, x(s))|(\zeta)$$

$$+ \frac{|v_1||\tau_2 - \tau_1|}{|\varLambda|} \Big(|\beta| \; {}^\rho I^p J^q |f(s, x(s))|(\xi) + J^q |f(s, x(s))|(T)\Big)$$

$$\leq \frac{L_1}{\Gamma(q)} \left| \int_0^{\tau_1} [(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]ds + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{q-1}ds \right|$$

$$+ \frac{L_1 |\alpha||v_2||\tau_2 - \tau_1|}{|\varLambda|} I_\eta^{\gamma,\delta} J^q(\zeta) + \frac{L_1 |v_1||\tau_2 - \tau_1|}{|\varLambda|} \Big(|\beta| \; {}^\rho I^p J^q(\xi) + J^q(T)\Big)$$

$$\leq \frac{L_1}{\Gamma(q+1)}(\tau_2^q - \tau_1^q) + \frac{L_1 |\alpha||v_2||\tau_2 - \tau_1|}{|\varLambda|} \frac{\zeta^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)}$$

$$+ \frac{L_1 |v_1||\tau_2 - \tau_1|}{|\varLambda|} \Bigg(\frac{1}{\Gamma(q+1)} \frac{\xi^{q+\rho p}}{\rho^p} \frac{\Gamma((q+\rho)/\rho)}{\Gamma((q+\rho p + \rho)/\rho)} + \frac{T^q}{\Gamma(q+1)} \Bigg).$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore by the Arzelá-Ascoli theorem the operator $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Next, we consider the set $V = \{x \in \mathcal{C} : x = \mu\mathcal{P}x, \; 0 < \mu < 1\}$. In order to show that V is bounded, let $x \in V$ and $t \in [0, T]$. Then

$$\|x\| \quad \leq \quad L_1 \Bigg\{ \frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|(|v_4| + T|v_3|)}{|\varLambda|} \frac{\zeta^q \Gamma(\gamma + (q/\eta) + 1)}{\Gamma(q+1)\Gamma(\gamma + (q/\eta) + \delta + 1)}$$

$$+ \frac{(|v_2| + T|v_1|)}{|\varLambda|} \Bigg(\frac{1}{\Gamma(q+1)} \frac{\xi^{q+\rho p}}{\rho^p} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho p+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)} \Bigg) \Bigg\} = M_1.$$

Therefore, V is bounded. Hence, by Lemma 7, problem (1.1) has at least one solution on $[0, T]$. □

3.6 Existence result via Leray-Schauder's Degree Theory

Theorem 6. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that*

(H_6) *there exist constants* $0 \le \nu < \Psi^{-1}$, *and* $M > 0$ *such that*

$$|f(t,x)| \le \nu|x| + M \quad for \ all \quad (t,x) \in [0,T] \times \mathbb{R},$$

where Ψ *is defined by (3.2).*

Then the boundary value problem (1.1) has at least one solution on $[0,T]$.

Proof.   Let us consider the operator equation

$$x = \mathcal{P}x, \tag{3.4}$$

where the operator $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ is defined by (3.1). Our proof will be complete once we establish the existence of at least one solution $x \in \mathcal{C}$ for (3.4). Set a ball $B_R \subset C[0,T]$ with a constant radius $R > 0$ as follows

$$B_R = \{x \in \mathcal{C} : \max_{t \in [0,T]} |x(t)| < R\}$$

and show that the operator $\mathcal{P} : \overline{B}_R \to C[0,T]$ satisfies the condition

$$x \ne \theta \mathcal{P}x, \quad \forall x \in \partial B_R, \quad \forall \theta \in [0,1]. \tag{3.5}$$

Define $H(\theta, x) = \theta \mathcal{P}x$, $x \in \mathcal{C}$, $\theta \in [0,1]$. We know from Theorem 2 that the operator $\mathcal{P}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_θ defined by $h_\theta(x) = x - H(\theta, x) = x - \theta \mathcal{P}x$ is completely continuous. If (3.5) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(h_\theta, B_R, 0) &= \deg(I - \theta\mathcal{P}, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \ne 0, \quad 0 \in B_R, \end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, we have $h_1(x) = x - \mathcal{P}x = 0$ for at least one $x \in B_R$. Let us assume that $x = \theta\mathcal{P}x$ for some $\theta \in [0,1]$ and for all $t \in [0,T]$ so that

$$\begin{aligned} |x(t)| &= |\theta\mathcal{P}x(t)| \\ &\le J^q|f(s,x(s))|(t) + \frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)I_\eta^{\gamma,\delta}J^q|f(s,x(s))|(\zeta) \\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta|\ {}^{\rho}I^p J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big) \\ &\le (\nu|x| + M)J^q p(s)(T) + (\nu|x| + M)\frac{|\alpha|}{|\Lambda|}(|v_4| + T|v_3|)I_\eta^{\gamma,\delta}J^q p(s)(\zeta) \\ &\quad + (\nu|x| + M)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\beta|\ {}^{\rho}I^p J^q p(s)(\xi) + J^q p(s)(T)\Big) \\ &= (\nu|x| + M)\Psi, \end{aligned}$$

which, on taking the norm $\sup_{t \in [0,T]} |x(t)| = \|x\|$ and solving for $\|x\|$, yields

$$\|x\| \le \frac{M\Psi}{1 - \nu\Psi}.$$

If $R = \dfrac{M\Psi}{1 - \nu\Psi} + 1$, the inequality (3.5) holds. This completes the proof. $\square$

4 Examples

In this section, we illustrate the results obtained in the last section.

Example 1. Consider the following fractional-order boundary value problem involving nonlocal Erdélyi-Kober and generalized Riemann-Liouville fractional integral conditions

$$
\begin{cases}
{}^{c}D^{\frac{5}{4}}x(t) = \dfrac{1}{15}\dfrac{e^{-2t}}{(t+3)^2}\dfrac{|x(t)|}{1+3|x(t)|} + \dfrac{1}{4}\log(t+1), & t \in [0, 5/2], \\
x(0) = \dfrac{2}{\sqrt{3}}I_{\frac{2}{3}}^{\frac{\sqrt{\pi}}{4},\frac{2}{e^2}}x\left(\dfrac{1}{2}\right), & x\left(\dfrac{5}{2}\right) = \dfrac{3}{14}I^{\frac{1}{\sqrt{2}}}_{\frac{5}{13}}x\left(\dfrac{3}{2}\right).
\end{cases}
\tag{4.1}
$$

Here $q = 5/4$, $T = 5/2$, $\alpha = 2/\sqrt{3}$, $\gamma = \sqrt{\pi}/4$, $\delta = 2/e^2$, $\eta = 2/3$, $\zeta = 1/2$, $\beta = 3/14$, $\rho = 5/13$, $p = 1/\sqrt{2}$, $\xi = 3/2$ and $f(t,x) = (1/15)(e^{-2t}/(t+3)^2)(|x|/(1+3|x|)) + (1/4)\log(t+1)$. Computing the given data, we obtain the following constants $v_1 = -0.122299372$, $v_2 = 0.4460979363$, $v_3 = 0.6773267311$, $v_4 = 2.206732908$ and $\Lambda = 0.0322720081 \neq 0$. Since $|f(t,x) - f(t,y)| \leq (1/135)|x - y|$, the condition (H_1) holds with $L = 1/135$. In addition, we find that $|f(t,x)| \leq e^{-2t}/(15(t+3)^2) + (1/4)\log(t+1)$ and $\Psi_1 = 0.9688798089 < 1$. Therefore, by applying Theorem 1, problem (4.1) has at least one solution on $[0, 5/2]$.

Example 2. Consider the following fractional-order boundary value problem involving nonlocal Erdélyi-Kober and generalized Riemann-Liouville fractional integral conditions

$$
\begin{cases}
{}^{c}D^{\frac{11}{6}}x(t) = \dfrac{(t^{\frac{1}{3}}+t^{\frac{1}{2}})}{15}\left(\dfrac{2x^2(t)+3|x(t)|}{1+|x(t)|} + 2\right), & t \in [0, 1/2], \\
x(0) = \dfrac{1}{\sqrt{\pi}}I_{\frac{5}{\pi^2}}^{\frac{4}{7},\frac{3}{\sqrt{10}}}x\left(\dfrac{1}{8}\right), & x\left(\dfrac{1}{2}\right) = \dfrac{\sqrt{7}}{12}I^{\frac{5}{7}}_{\frac{11}{18}}x\left(\dfrac{1}{4}\right).
\end{cases}
\tag{4.2}
$$

Here $q = 11/6$, $T = 1/2$, $\alpha = 1/\sqrt{\pi}$, $\gamma = 4/7$, $\delta = 3/\sqrt{10}$, $\eta = 5/\pi^2$, $\zeta = 1/8$, $\beta = \sqrt{7}/12$, $\rho = 11/18$, $p = 5/7$, $\xi = 1/4$ and $f(t,x) = ((t^{1/3}+t^{1/2})/15)(((2x^2+3|x|)/(1+|x|)) + 2)$. By direct computation of given constants, we have $v_1 = 0.6273547690$, $v_2 = 0.02136672311$, $v_3 = 0.8527688407$, $v_4 = 0.4777750041$ and $\Lambda = 0.3179553030 \neq 0$. Choosing $p(t) = (t^{\frac{1}{3}} + t^{\frac{1}{2}})/15$ and $\Phi(|x|) = 2|x| + 5$, we have $A_1 = 0.0001321467571$ and $A_2 = 0.01053345765$. Then there exists a positive constant $N > 0.1070970081$ satisfying inequality in (H_4). Therefore, all conditions of Theorem 2 are fulfilled. Hence, problem (4.2) has at least one solution on $[0, 1/2]$.

Example 3. Consider the following fractional-order boundary value problem involving nonlocal Erdélyi-Kober and generalized Riemann-Liouville fractional integral conditions

$$
\begin{cases}
{}^{c}D^{\frac{3}{2}}x(t) = \dfrac{1}{8}\left(\dfrac{x^2(t)+4|x(t)|}{3+|x(t)|}\right)\cos^2 t + \dfrac{1}{2\sin^2 t}, & t \in [0, 3/2], \\
x(0) = \dfrac{3}{7}I_{\frac{\sqrt{5}}{9}}^{\frac{1}{2},\frac{2}{\sqrt{3}}}x\left(\dfrac{1}{2}\right), & x\left(\dfrac{3}{2}\right) = \dfrac{4}{5}I^{\frac{6}{11}}_{\frac{3}{7}}x\left(\dfrac{11}{8}\right).
\end{cases}
\tag{4.3}
$$

Here $q = 3/2$, $T = 3/2$, $\alpha = 3/7$, $\gamma = 1/2$, $\delta = 2/\sqrt{3}$, $\eta = \sqrt{5}/9$, $\zeta = 1/2$, $\beta = 4/5$, $\rho = 3/7$, $p = 6/11$, $\xi = 11/8$ and $f(t, x) = (1/8)((x^2 + 4|x|)/(3 + |x|)) \cos^2 t + (1/2) \sin^2 t$. Using the given data, we find that $v_1 = 0.7451980583$, $v_2 = 0.02931419848$, $v_3 = -0.910393214$, $v_4 = 0.487833885$, $\Lambda = 0.3368454165$ and $\Psi = 5.736302996$. Since $|f(t, x) - f(t, y)| \leq (1/6)|x - y|$, the condition (H_1) is satisfied with $L = 1/6$. Obviously $L\Psi = 0.9560504993 < 1$. Thus, by Theorem 3, problem (4.3) has a unique solution on $[0, 3/2]$.

Example 4. Consider the following fractional-order boundary value problem

$$
\begin{cases}
{}^c D^{\frac{7}{5}} x(t) = \dfrac{1}{3(t+2)^3} \left(\sin |x(t)| + \dfrac{|x(t)|}{1+|x(t)|} \right) + \dfrac{1}{2}(t^{\frac{1}{4}}+1), \ t \in [0, 2], \\[3mm]
x(0) = \dfrac{\sqrt{2}}{5} I^{\frac{\pi}{2}, \frac{1}{4}}_{\frac{3}{7}} x \left(\dfrac{1}{2} \right), \quad x(2) = \dfrac{\sqrt{3}}{4} \, {}_{\frac{8}{15}} I^{\frac{13}{19}} x \left(\dfrac{3}{2} \right),
\end{cases}
\tag{4.4}
$$

where $q = 7/5$, $T = 2$, $\alpha = \sqrt{2}/5$, $\gamma = \pi/2$, $\delta = 1/4$, $\eta = 3/7$, $\zeta = 1/2$, $\beta = \sqrt{3}/4$, $\rho = 8/15$, $p = 13/19$, $\xi = 3/2$ and $f(t, x) = (1/(3(t+2)^3))(\sin |x| + (|x|)/(1 + |x|)) + (1/2)(t^{\frac{1}{4}} + 1)$. Using the given values, we find that $v_1 = 0.7681027408$, $v_2 = 0.09689608493$, $v_3 = 0.2722751129$, $v_4 = 1.416898935$, $\Lambda = 1.114706347$. Choosing $z(t) = t^{\frac{1}{3}}/12$, we get $A^* = 0.4296509478$. Clearly the condition (H_2) holds true as

$$
|f(t, x) - f(t, y)| \leq \frac{t^{\frac{1}{3}}}{12} \left(\frac{|x - y|}{0.4296509478 + |x - y|} \right).
$$

Therefore, from Theorem 4, problem (4.4) has a unique solution on $[0, 2]$.

Example 5. Consider the following problem of Caputo fractional differential equation supplemented with nonlocal Erdélyi-Kober and generalized Riemann-Liouville fractional integral conditions

$$
\begin{cases}
{}^c D^{\frac{27}{17}} x(t) = \dfrac{1}{2} \tan^{-1} \left(\dfrac{x^2(t) + |x(t)|}{1 + |x(t)|} \right) (3 \cos^2 t + 1) + \pi, \ t \in \left[0, \dfrac{5}{2} \right], \\[3mm]
x(0) = \dfrac{1}{\sqrt{7}} I^{\frac{5}{8}, \frac{7}{13}}_{\frac{5}{9}} x(1), \quad x \left(\dfrac{5}{2} \right) = \dfrac{1}{\sqrt{3}} \, {}_{\frac{4}{3}} I^{\frac{\sqrt{7}}{6}} x(2),
\end{cases}
\tag{4.5}
$$

where $q = 27/17$, $T = 5/2$, $\alpha = 1/\sqrt{7}$, $\gamma = 5/8$, $\delta = 7/13$, $\eta = 5/9$, $\zeta = 1$, $\beta = 1/\sqrt{3}$, $\rho = 4/3$, $p = \sqrt{7}/6$, $\xi = 2$ and $f(t, x) = (1/2) \tan^{-1}((x^2 + |x|)/(1 + |x|))(3 \cos^2 t + 1) + \pi$. By the given data, we find that $v_1 = 0.6863802809$, $v_2 = 0.2019299623$, $v_3 = 0.1702153301$, $v_4 = 1.218454775$, $\Lambda = 0.8706949059 \neq 0$. It is clear that $|f(t, x)| \leq 2\pi$. Thus all the assumptions of Theorem 5 are satisfied. In consequence, problem (4.5) has at least one solution on $[0, 5/2]$.

Example 6. Consider the following nonlocal nonlinear fractional-order problem involving Erdélyi-Kober and generalized Riemann-Liouville fractional integral

conditions

$$
\begin{cases}
{}^{c}D^{\frac{43}{26}}x(t) = \dfrac{(\sin^2 t + 1)}{120}\left(\dfrac{x^2(t)}{1 + |x(t)|}\right)e^{-t} + \dfrac{2|x(t)|}{1 + 3|x(t)|} + \dfrac{1}{3},\ t \in [0,3]\,, \\[3mm]
x(0) = \dfrac{3}{16}I^{\frac{4}{\sqrt{5}},\frac{\sqrt{3}}{2}}_{\frac{1}{\sqrt{6}}}x\left(\dfrac{3}{2}\right),\quad x(3) = \dfrac{2}{\sqrt{5}}I^{\frac{3}{7}}_{\frac{\sqrt{e}}{2}}x\left(\dfrac{5}{2}\right).
\end{cases}
$$

$$(4.6)$$

Here $q = 43/26$, $T = 3$, $\alpha = 3/16$, $\gamma = 4/\sqrt{5}$, $\delta = \sqrt{3}/2$, $\eta = 1/\sqrt{6}$, $\zeta = 3/2$, $\beta = 2/\sqrt{5}$, $\rho = \sqrt{e}/2$, $p = 3/7$, $\xi = 5/2$ and $f(t,x) = ((\sin^2 t + 1)/120)(x^2/(1 + |x|))e^{-t} + ((2|x|)/(1 + 3|x|)) + (1/3)$. With the given values, we have that $v_1 = 0.9213181351$, $v_2 = 0.06775588797$, $v_3 = -0.658410315$, $v_4 = 0.474997666$, $\Lambda = 0.3930127883 \neq 0$ and $\Psi = 55.66868159$. Also $|f(t,x)| \leq (1/60)|x| + 1$ with $\nu = (1/60) = 0.01666666667 < 0.01796342165 = \Psi^{-1}$ and $M = 1$. Therefore, the conclusion of Theorem 6 implies that problem (4.6) has at least one solution on $[0,3]$.

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