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A Weighted Discrete Universality Theorem for Periodic Zeta-Functions. II

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Abstract. In the paper, a weighted theorem on the approximation of a wide class of analytic functions by shifts $\zeta(s+ik^\alpha h;\mathfrak{a})$, $k\in\mathbb{N}$, $0<\alpha<1$, and $h>0$, of the periodic zeta-function $\zeta(s;\mathfrak{a})$ with multiplicative periodic sequence $\mathfrak{a}$, is obtained.

Keywords: Hurwitz zeta-function, Mergelyan theorem, periodic zeta-function, universality.

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1 Introduction

Let $s=\sigma+it$ be a complex variable, and $\mathfrak{a}=\{a_m : m\in\mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q\in\mathbb{N}$. The periodic zeta-function $\zeta(s;\mathfrak{a})$ is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m}{m^s}.$$

Moreover, the function $\zeta(s;\mathfrak{a})$ is meromorphically continued to the whole complex plane. Really, let $\zeta(s,\alpha)$ denote the Hurwitz zeta-function with parameter α, $0<\alpha\leqslant 1$, which, for $\sigma>1$, is given by the series

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s}$$

and has the meromorphic continuation to the whole complex plane with unique simple pole at the point $s=1$ with residue 1. Since, in virtue of periodicity of the sequence $\mathfrak{a}$,

$$\zeta(s;\mathfrak{a}) = \frac{1}{q^s}\sum_{m=1}^{q} a_m \zeta\left(s, \frac{m}{q}\right), \quad \sigma > 1, \tag{1.1}$$

we see that the function $\zeta(s;\mathfrak{a})$ is meromorphic in the whole complex plane with unique simple pole at the point $s = 1$ with residue

$$r = \frac{1}{q}\sum_{m=1}^{q} a_m.$$

If $r = 0$, then the function $\zeta(s;\mathfrak{a})$ is entire. If $a_m = 1$, for all $m \in \mathbb{N}$, then $\zeta(s;\mathfrak{a})$ becomes the Riemann zeta-function $\zeta(s)$,

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1.$$

Therefore, the investigation of the function $\zeta(s;\mathfrak{a})$ is a modern problem of analytic number theory.

In [24], S.M. Voronin discovered the universality of the Riemann zeta-function. The Voronin theorem, roughly speaking, asserts that a wide class of analytic functions in a certain region can be approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. Later, it turned out that some other zeta and L-functions, including the function $\zeta(s;\mathfrak{a})$, are also universal in the Voronin sense. The first universality results for $\zeta(s;\mathfrak{a})$ were obtained in [1], [2], [21] and [22]. The universality of $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ was considered in [16], [23], [18] and [17]. We remind the paper [6], where a new type of universality for the function $\zeta(s;\mathfrak{a})$ was introduced. Joint universality theorems for periodic zeta-functions were proved in [5], [10], [11], [12], [13], [14] and [15].

In [8], a weighted universality theorem for the Riemann zeta-function was obtained. Generalizations of a theorem of such a type were given in [9] and [4]. The weighted universality for the function $\zeta(s;\mathfrak{a})$ was began to study in [18]. We remind the main result of [18]. Let $\hat{w}(t)$ be a positive function of bounded variation on $[T_0, \infty]$, $T_0 > 0$, such that the variation $V_a^b \hat{w}$ on $[a, b]$ satisfies the inequality $V_a^b \hat{w} \leqslant c\hat{w}(a)$, $c > 0$, for any $[a, b] \subset [T_0, \infty)$. Define

$$U = U(T, \hat{w}) = \int_{T_0}^{T} \hat{w}(t)\,\mathrm{d}\,t$$

and suppose that $\lim_{T\to\infty} U(T, \hat{w}) = +\infty$. Let $\mathcal{K}$ be the class of compact subsets of the strip $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$ with connected complements, and let $H_0(K)$, $K \in \mathcal{K}$, be the class of continuous non-vanishing functions on K which are analytic in the interior of K. Moreover, let I_A denote the indicator function of the set A. We remind that the sequence $\mathfrak{a} = \{a_m\}$ is called multiplicative if $a_{mn} = a_m a_n$ for all coprimes $m, n \in \mathbb{N}$. Now we state an universality theorem from [18].

Theorem 1. *Suppose that the weight function $\hat{w}(t)$ satisfies all above conditions, the sequence $\mathfrak{a}$ is multiplicative and*

$$\sum_{l=1}^{\infty} \frac{|a_{p^l}|}{p^{\frac{l}{2}}} \leqslant c < 1$$

for all primes p. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} \hat{w}(\tau) I_{\left\{\tau: \sup\limits_{s\in K} |\zeta(s+i\tau;\mathfrak{a})-f(s)|<\varepsilon\right\}}(\tau)\,\mathrm{d}\,\tau > 0.$$

In [17], a discrete version of Theorem 1 was obtained. In discrete universality theorems, τ in shifts $\zeta(s+i\tau;\mathfrak{a})$ takes values from a certain discrete set. In [17], an arithmetic progression $\{kh : k \in \mathbb{N}\}$, $h > 0$, was used. Let $w(u)$ be a non-increasing positive function having a continuous derivative such that, for $h > 0$, $w(u) \ll_h w(hu)$ and $(w'(u))^2 \ll w(u)$. Define

$$V = V(N, w) = \sum_{k=1}^{N} w(k)$$

and suppose that $\lim_{N\to\infty} V(N, w) = +\infty$ as $N \to \infty$. Moreover, let

$$L(\mathbb{P}, h, \pi) = \left\{(\log p : p \in \mathbb{P}), \frac{\pi}{h}\right\},$$

where $\mathbb{P}$ is the set of all prime numbers. Then the following weighted discrete universality theorem is true.

Theorem 2. *Suppose that the function $w(u)$ satisfies all above hypotheses, the sequence $\mathfrak{a}$ is the same as in Theorem 1, and the set $L(\mathbb{P}, h, \pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{V} \sum_{k=1}^{N} w(k) I_{\left\{k: \sup\limits_{s\in K} |\zeta(s+ikh;\mathfrak{a})-f(s)|<\varepsilon\right\}}(k) > 0.$$

It is not difficult to see that the function $w(u) = \frac{1}{u}$ satisfies the hypotheses of Theorem 2. Since e^{π} is transcendental number, the set $L(\mathbb{P}, h, \pi)$ with rational h is linearly independent over $\mathbb{Q}$.

The aim of this paper is to prove an analogue of Theorem 2 for the discrete set $\{k^{\alpha}h : k \in \mathbb{N}\}$ with fixed $0 < \alpha < 1$.

Theorem 3. *Suppose that the function $w(u)$ has a continuous derivative $w'(u)$ for $u \geqslant 1$ such that*

$$\int_1^N u\,|w'(u)|\,\mathrm{d}\,u \ll V,$$

and $\mathfrak{a}$ is the same as in Theorem 2. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$ and $h > 0$,

$$\liminf_{N\to\infty} \frac{1}{V} \sum_{k=1}^{N} w(k) I_{\left\{1\leqslant l\leqslant N: \sup\limits_{s\in K} |\zeta(s+il^{\alpha}h;\mathfrak{a})-f(s)|<\varepsilon\right\}}(k) > 0.$$

Differently from Theorem 2, we do not require the linear independence over $\mathbb{Q}$ of the set $L(\mathbb{P}, h, \pi)$.

2 The main lemma

Let $H(D)$ denote the space of analytic functions on D endowed with the topology of uniform convergence on compacta, and let $\mathcal{B}(X)$ stand for the Borel σ-field of the space X. For the proof of Theorem 3, we will apply the weak convergence of probability measures on $(H(D), \mathcal{B}(H(D)))$. We start with a limit theorem for probability measures on $(\Omega, \mathcal{B}(\Omega))$, where

$$\Omega = \prod_p \gamma_p,$$

and $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. By the Tikhonov theorem, the torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Thus, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and this leads to the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the circle γ_p, $p \in \mathbb{P}$. For $A \in \mathcal{B}(\Omega)$, define

$$Q_{N,w}(A) = \frac{1}{V} \sum_{k=1}^{N} w(k) I_{\hat{A}}(k),$$

where, for brevity, $\hat{A} = \big\{1 \leqslant l \leqslant N : \big(p^{-il^\alpha h} : p \in \mathbb{P}\big) \in A\big\}$.

For the investigation of $Q_{N,w}$, we will apply the notion of sequences uniformly distributed modulo 1. We remind that a sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for every interval $I = [a, b) \subset [0, 1)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} I_I(\{x_k\}) = b - a,$$

where $\{x_k\}$ denotes the fractional part of x_k. For us, the Weyl criterion, see, for example, [7], which states that a sequence $\{x_k\}$ is uniformly distributed modulo 1 if and only if, for all $m \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i x_k m} = 0,$$

will be useful.

Lemma 1. *Suppose that the function $w(t)$ has a continuous derivative such that $\int_1^N u|w'(u)|\,\mathrm{d}\,u \ll U$ for $t \geqslant 1$ and α, $0 < \alpha < 1$, is a fixed number. Then $Q_{N,w}$ converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We consider the Fourier transform $g_{N,w}(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z},\ p \in \mathbb{P})$ of $Q_{N,w}$, i.e.,

$$g_{N,w}(\underline{k}) = \int_\Omega \prod_p \omega^{k_p}(p)\,\mathrm{d}\,Q_{N,w},$$

where only a finite number of integers k_p are distinct from zero. By the definition of $Q_{N,w}$, we find that

$$\begin{aligned} g_{N,w}(\underline{k}) &= \frac{1}{V}\sum_{k=1}^{N} w(k)\prod_p p^{-ik^\alpha h k_p} \\ &= \frac{1}{V}\sum_{k=1}^{N} w(k)\exp\left\{-ik^\alpha h\sum_p k_p\log p\right\}, \end{aligned} \tag{2.1}$$

where only a finite number of integers k_p are distinct from zero. Clearly, by (2.1),

$$g_{N,w}(\underline{0}) = 1. \tag{2.2}$$

Now suppose that $\underline{k} \neq \underline{0}$. Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$, we have that

$$\sum_p k_p \log p \neq 0.$$

It is known, [7, Exercise 3.10], that the sequence $\{ak^\alpha : k \in \mathbb{N}\}$ with $0 < \alpha < 1$ and $a \neq 0$ is uniformly distributed modulo 1. Therefore,

$$R(u) \stackrel{def}{=} \sum_{k\leqslant u}\exp\left\{-ik^\alpha h\sum_p k_p\log p\right\} = o(u)$$

as $u \to \infty$. Hence, using (2.1) and summing by parts, we find that

$$\begin{aligned} g_{N,w}(\underline{k}) &= \frac{R(N)w(N)}{V} - \frac{1}{V}\int_1^N R(u)w'(u)\,\mathrm{d}\,u \\ &= o\left(\frac{Nw(N)}{V}\right) + o\left(\frac{1}{V}\int_1^N u|w'(u)|\,\mathrm{d}\,u\right) = o(1) \end{aligned}$$

as $N \to \infty$, since

$$Nw(N) = V + \int_1^N u|w'(u)|\,\mathrm{d}\,u \ll V.$$

This together with (2.2) gives

$$\lim_{T\to\infty} g_{T,w}(\underline{k}) = \begin{cases} 1, & \text{if} \quad \underline{k} = \underline{0}, \\ 0, & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases} \tag{2.3}$$

Since the right-hand side of (2.3) is the Fourier transform of the Haar measure m_H, by a continuity theorem for probability measures on compact groups, we obtain that $Q_{N,w}$ converges weakly to m_H as $N \to \infty$.

3 A limit theorem

We remind that $H(D)$ is the space of analytic functions on $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \omega; \mathfrak{a})$ by the formula

$$\zeta(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s},$$

where

$$\omega(m) = \prod_{p^l || m} \omega^l(p), \quad m \in \mathbb{N},$$

and $p^l \,||\, m$ denotes that $p^l \mid m$ but $p^{l+1} \nmid m$. Note that the latter series, for almost all $\omega \in \Omega$, is uniformly convergent on compact subsets of the strip D. Moreover, for almost all $\omega \in \Omega$, the equality

$$\zeta(s, \omega; \mathfrak{a}) = \prod_p \left(1 + \sum_{l=1}^{\infty} \frac{a_{p^l} \omega^l(p)}{p^{ls}}\right)$$

holds. Denote by P_ζ the distribution of the random element $\zeta(s, \omega; \mathfrak{a})$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega; \mathfrak{a}) \in A), \quad A \in \mathcal{B}(H(D)).$$

Let, for $A \in \mathcal{B}(H(D))$,

$$P_{N,w}(A) = \frac{1}{V} \sum_{k=1}^{N} w(k) I_{\{1 \leqslant l \leqslant N : \zeta(s + il^\alpha h; \mathfrak{a}) \in A\}}(k).$$

Theorem 4. *Suppose that the function $w(t)$ and the sequence $\mathfrak{a}$ satisfy hypotheses of Theorem 3. Then $P_{N,w}$ converges weakly to P_ζ as $N \to \infty$. Moreover, the support of the measure P_ζ is the set $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.*

We divide the proof of Theorem 4 into few lemmas. The first of them is a weighted limit theorem for absolutely convergent Dirichlet series. Let $\theta > \frac{1}{2}$ be a fixed number, and, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^\theta\right\}.$$

Define two series

$$\zeta_n(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^s} \qquad \text{and} \qquad \zeta_n(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m) v_n(m)}{m^s},$$

which are absolutely convergent [16] for $\sigma > \frac{1}{2}$. Consider the function $u_n : \Omega \to H(D)$ defined by the formula

$$u_n(\omega) = \zeta_n(s, \omega; \mathfrak{a}).$$

Since the series for $\zeta_n(s,\omega;\mathfrak{a})$ is absolutely convergent for $\sigma > \frac{1}{2}$, the function u_n is continuous one. Let $R_n = m_H u_n^{-1}$, where

$$R_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1}A), \quad A \in \mathcal{B}(H(D)),$$

and let, for $A \in \mathcal{B}(H(D))$,

$$P_{T,n,w}(A) = \frac{1}{V}\sum_{k=1}^{N} w(k) I_{\{1\leqslant l\leqslant N: \zeta_n(s+il^\alpha h;\mathfrak{a})\in A\}}(k).$$

Lemma 2. *Suppose that the function $w(t)$ and the sequence $\mathfrak{a}$ are the same as in Theorem 3. Then $P_{N,n,w}$ converges weakly to R_n as $N\to\infty$.*

Proof. The lemma is derived from Lemma 1 in the same way as Lemma 2 in [17].

The next lemma deals with the approximation of $\zeta(s;\mathfrak{a})$ by $\zeta_n(s;\mathfrak{a})$. Denote by ρ the metric in $H(D)$, see, for example, [18].

Lemma 3. *Suppose that the function $w(t)$ and the sequence $\mathfrak{a}$ satisfy the hypotheses of Theorem 3. Then the equality*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{V}\sum_{k=1}^{N} w(k)\rho(\zeta(s+ik^\alpha h;\mathfrak{a}), \zeta_n(s+ik^\alpha h;\mathfrak{a})) = 0$$

is true.

Proof. For the same θ as above and $n\in\mathbb{N}$, define

$$l_n(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right) n^s,$$

where $\Gamma(s)$ is the Euler gamma-function. Then, for $\theta<\sigma<1$, the representation [16]

$$\begin{aligned}\zeta_n(s;\mathfrak{a}) &= \frac{1}{2\pi i}\int_{\theta-\sigma-i\infty}^{\theta-\sigma+i\infty}\zeta(s+z;\mathfrak{a}) l_n(z)\frac{\mathrm{d}z}{z}\\ &= \zeta(s;\mathfrak{a}) + \operatorname*{Res}_{z=1-s}\zeta(s+z;\mathfrak{a})\frac{l_n(z)}{z}\end{aligned} \tag{3.1}$$

holds. Using equality (1.1) and the estimate

$$\int_1^T |\zeta(\sigma+it,\alpha)|^2\,\mathrm{d}t \ll T, \quad \frac{1}{2}<\sigma<1,$$

we find that, for $\frac{1}{2}<\sigma<1$, and $\tau\in\mathbb{R}$,

$$\int_1^T |\zeta(\sigma+it+i\tau;\mathfrak{a})|^2\,\mathrm{d}t \ll T(1+|\tau|) \tag{3.2}$$

and, by the Cauchy integral formula,

$$\int_1^T \left|\zeta'(\sigma+it+i\tau;\mathfrak{a})\right|^2 \mathrm{d}\,t \ll T(1+|\tau|). \tag{3.3}$$

It is not difficult to see that, for $2 \leqslant k \leqslant N$,

$$(k+1)^\alpha - k^\alpha \geqslant \frac{\alpha}{2N^{1-\alpha}}.$$

Therefore, the Gallagher lemma, see [20, Lemma 1.4], together with estimates (3.2) and (3.3) yields, for $\frac{1}{2} < \sigma < 1$ and $\tau \in \mathbb{R}$,

$$\begin{aligned}\sum_{k=1}^{N}|\zeta(\sigma+ik^\alpha h+i\tau;\mathfrak{a})|^2 &\ll N^{1-\alpha}\int_1^{N^\alpha h}|\zeta(\sigma+it+i\tau;\mathfrak{a})|^2\,\mathrm{d}\,t\\ &+\left(\int_1^{N^\alpha h}|\zeta(\sigma+it+i\tau;\mathfrak{a})|^2\,\mathrm{d}\,t\int_1^{N^\alpha h}|\zeta'(\sigma+it+i\tau;\mathfrak{a})|^2\,\mathrm{d}\,t\right)^{1/2}\\ &= N(1+|\tau|).\end{aligned}$$

Hence, for the same σ and τ,

$$\begin{aligned}&\sum_{k=1}^{N} w(k)|\zeta(s+ik^\alpha h+i\tau;\mathfrak{a})|^2\\ &\quad\ll w(N)\sum_{k=1}^{N}|\zeta(s+ik^\alpha h+i\tau;\mathfrak{a})|^2+\int_1^N|\zeta(\sigma+k^\alpha h+i\tau;\mathfrak{a})|^2|w'(u)|\,\mathrm{d}\,u\\ &\quad\ll Nw(N)(1+|\tau|)+(1+|\tau|)\int_1^N u|w'(u)|\,\mathrm{d}\,u \ll V(1+|\tau|).\end{aligned} \tag{3.4}$$

Now let K be a compact subset of the strip D. Then equality (3.1), the Cauchy integral formula and (3.4) show that

$$\begin{aligned}&\frac{1}{V}\sum_{k=1}^{N} w(k)\sup_{s\in K}|\zeta(s+ik^\alpha h;\mathfrak{a})-\zeta_n(s+ik^\alpha h;\mathfrak{a})|\\ &\quad\ll\int_{-\infty}^{\infty}|l_n(\sigma_1+it)|(1+|t|)\ \mathrm{d}\,t+o(1)\end{aligned}$$

as $N\to\infty$ with some $\sigma_1<0$. This, the definitions of $l_n(s)$ and the metric ρ prove the lemma.

Proof of Theorem 4. On a certain probability space $(\hat{\Omega},\mathcal{A},\mu)$, define the random variable θ_N by the formula

$$\mu(\theta_N = k^\alpha h) = \frac{w(k)}{V}, \quad k=1,\dots,N.$$

Let

$$X_{N,n,w} = X_{N,n,w}(s) = \zeta_n(s+i\theta_N;\mathfrak{a}),$$

and let X_n be the $H(D)$-valued random element having the distribution R_n, where R_n is the probability measure from Lemma 2. Thus, denoting by $\xrightarrow{\mathcal{D}}$ the convergence in distribution, we may to rewrite the assertion of Lemma 2 in the form

$$X_{N,n,w} \xrightarrow[N\to\infty]{\mathcal{D}} X_n. \tag{3.5}$$

Now we will consider the family of probability measures $\{R_n : n \in \mathbb{N}\}$, and we will prove that this family is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D)$ such that

$$R_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. The series for $\zeta_n(s;\mathfrak{a})$ and $\zeta_n'(s;\mathfrak{a})$ are absolutely convergent for $\sigma > \frac{1}{2}$, thus

$$\limsup_{T\to\infty} \frac{1}{T}\int_1^T |\zeta_n(\sigma+it;\mathfrak{a})|^2 \,\mathrm{d}\,t = \sum_{m=1}^{\infty} \frac{|a_m|^2 v_n^2(m)}{m^{2\sigma}} \leqslant \sum_{m=1}^{\infty} \frac{|a_m|^2}{m^{2\sigma}} \leqslant C < \infty$$

and

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T}\int_1^T |\zeta_n'(\sigma+it;\mathfrak{a})|^2 \,\mathrm{d}\,t &= \sum_{m=1}^{\infty} \frac{|a_m|^2 v_n^2(m)\log^2 m}{m^{2\sigma}} \\ &\leqslant \sum_{m=1}^{\infty} \frac{|a_m|^2 \log^2 m}{m^{2\sigma}} \leqslant C' < \infty.\end{aligned}$$

Hence, using the Gallagher lemma, we find as above that, for $\sigma > \frac{1}{2}$,

$$\begin{aligned}\sum_{k=1}^{N} |\zeta_n(\sigma+ik^\alpha h;\mathfrak{a})|^2 &\ll N^{1-\alpha}\int_1^{N^\alpha h} |\zeta_n(\sigma+it;\mathfrak{a})|^2 \,\mathrm{d}\,t \\ &+ \left(\int_1^{N^\alpha h} |\zeta_n(\sigma+it;\mathfrak{a})|^2 \,\mathrm{d}\,t \int_1^{N^\alpha h} |\zeta_n'(\sigma+it;\mathfrak{a})|^2 \,\mathrm{d}\,t\right)^{1/2} \ll N.\end{aligned}$$

Therefore, by properties of the weight function $w(u)$, we obtain that, for $\sigma > \frac{1}{2}$,

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{V}\sum_{k=1}^{N} w(k)|\zeta_n(\sigma+it;\mathfrak{a})| \leqslant C < \infty. \tag{3.6}$$

Now let $\{K_l : l \in \mathbb{N}\} \subset D$ be a sequence of compact subsets which defines the metric ρ, see [18]. Then, using (3.6) and the Cauchy integral formula, we find that

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{V}\sum_{k=1}^{N} w(k) \sup_{s\in K_l} |\zeta_n(\sigma+it;\mathfrak{a})| \leqslant C_l < \infty.$$

We fix $\varepsilon > 0$ and define $M_l = M_l(\varepsilon) = 2^l C_l \varepsilon^{-1}$. Then, by the definition of $X_{N,n,w}$,

$$\limsup_{T\to\infty} \mu\left(\sup_{s\in K_l} |X_{N,n,w}(s)| > M_l\right)$$
$$= \limsup_{N\to\infty} \frac{1}{V}\sum_{k=1}^{N} w(k) I_{\left\{k:\ \sup\limits_{s\in K_l} |\zeta_n(s+ik^\alpha h;\mathfrak{a})|>M_l\right\}}(k)$$
$$\leqslant \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M_l V}\sum_{k=1}^{N} w(k) \sup_{s\in K_l} |\zeta_n(s+ik^\alpha h;\mathfrak{a})| \leqslant \frac{\varepsilon}{2^l}.$$

From this and (3.5), we deduce that, for all n, $l \in \mathbb{N}$,

$$\mu\left(\sup_{s\in K_l} |X_n(s)| > M_l\right) \leqslant \frac{\varepsilon}{2^l}. \tag{3.7}$$

The set $H_\varepsilon = \left\{g \in H(D) :\ \sup_{s\in K_l} |g(s)| \leqslant M_l,\ l \in \mathbb{N}\right\}$ is compact in the space $H(D)$, and, in view of (3.7),

$$\mu(X_n(s) \in H_\varepsilon) \geqslant 1 - \varepsilon \sum_{l=1}^{\infty} \frac{1}{2^l} \geqslant 1 - \varepsilon.$$

Hence, by the definition of X_n, for all $n \in \mathbb{N}$,

$$R_n(H_\varepsilon) \geqslant 1 - \varepsilon,$$

i.e., the family $\{R_n : n \in \mathbb{N}\}$ is tight. Therefore, by the Prokhorov theorem [3], it is relatively compact. Thus, every subsequence of $\{R_n\}$ have a subsequence $\{R_{n_r}\}$ weakly convergent to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $r \to \infty$. In other words,

$$X_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{3.8}$$

An application of Lemma 3 shows that, for $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V}\sum_{k=1}^{N} w(k) I_{\{k:\rho(\zeta(s+ik^\alpha h;\mathfrak{a}),\zeta_n(s+ik^\alpha h,\mathfrak{a}))\geqslant\varepsilon\}}(k)$$
$$\leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{V\varepsilon}\sum_{k=1}^{N} w(k)\rho(\zeta(s+ik^\alpha h;\mathfrak{a}),\zeta_n(s+ik^\alpha h,\mathfrak{a})) = 0. \tag{3.9}$$

Now, in view of relations (3.5), (3.8) and (3.9), we can apply Theorem 4.2 of [3] which shows that

$$\zeta(s+i\theta_N;\mathfrak{a}) \xrightarrow[N\to\infty]{\mathcal{D}} P.$$

This means that $P_{N,w}$ converges weakly to P as $N \to \infty$. Moreover, this shows that the measure P is independent of the subsequence $\{R_{n_r}\}$. This remark together with relative compactness of $\{R_n\}$ implies the relation

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P.$$

Consequently, by the definition of X_n, we have that R_n converges weakly to P as $n \to \infty$, i.e., $P_{N,w}$ as $N \to \infty$ converges weakly to the limit measure of R_n as $n \to \infty$. However, it is known [16] that

$$\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \zeta(s+i\tau;\mathfrak{a}) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

with multiplicative $\mathfrak{a}$, as $T \to \infty$, also converges weakly to the limit measure P of R_n, P coincides with P_ζ, and the support of P_ζ is the set S. Therefore, $P_{N,w}$ also converges weakly to P_ζ as $N \to \infty$.

4 Proof of universality

A proof of Theorem 3 is standard based on Theorem 4 and the Mergelyan theorem on the approximation of analytic functions by polynomials [19].

Proof of Theorem 4. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}\left|f(s) - e^{p(s)}\right| < \frac{\varepsilon}{2}. \tag{4.1}$$

Define the set

$$G_\varepsilon = \left\{g \in H(D) : \sup_{s\in K}\left|g(s) - e^{p(s)}\right| < \frac{\varepsilon}{2}\right\}.$$

Then the set G_ε is an open neighbourhood of the function $e^{p(s)}$ which, by Theorem 4, is an element of the support of P_ζ. Thus,

$$P_\zeta(G_\varepsilon) > 0. \tag{4.2}$$

Moreover, by Theorem 4 and the equivalent of weak convergence of probability measures in terms of open sets, we have that

$$\liminf_{N\to\infty} P_{N,w}(G_\varepsilon) \geqslant P_\zeta(G_\varepsilon).$$

This, (4.2) and the definitions of $P_{N,w}$ and G_ε show that

$$\liminf_{N\to\infty}\frac{1}{V}\sum_{k=1}^{N} w(k) I_{\left\{k:\sup_{s\in K}\left|\zeta(s+ik^\alpha h;\mathfrak{a})-e^{p(s)}\right|<\frac{\varepsilon}{2}\right\}}(k) > 0. \tag{4.3}$$

However, in view of (4.1),

$$\left\{k : \sup_{s\in K}\left|\zeta(s+ik^\alpha h;\mathfrak{a}) - e^{p(s)}\right| < \frac{\varepsilon}{2}\right\} \subset \left\{k : \sup_{s\in K}\left|\zeta(s+ik^\alpha h;\mathfrak{a}) - f(s)\right| < \varepsilon\right\}.$$

Therefore, the theorem follows from (4.3).

References

[1] B. Bagchi. *The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series.* Ph. D. Thesis, Indian Statist. Institute, Calcutta, 1981.

[2] B. Bagchi. A joint universality theorem for Dirichlet L-functions. *Mathematische Zeitschrift*, **181**(3):319–334, 1982.

[3] P. Billingsley. *Convergence of Probability Measures.* Wiley, New York, 1968.

[4] V. Garbaliauskienė. A weighted universality theorem for zeta-functions of elliptic curves. *Liet. matem. rink*, **45**(Spec. Issue):25–29, 2005.

[5] R. Kačinskaitė and A. Laurinčikas. The joint distribution of periodic zeta-functions. *Stud. Sci. Math. Hungarica*, **48**(2):257–279, 2011. https://doi.org/10.1556/SScMath.48.2011.2.1162.

[6] J. Kaczorowski. Some remarks on the universality of periodic L-functions. In R. Steuding and J. Steuding(Eds.), *New Directions in Value-Distribution Theory of Zeta and L-Functions*, pp. 113–120, Aachen, 2009. Shaker Verlag.

[7] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences.* Wiley, New York, 1979.

[8] A. Laurinčikas. On the universality of the Riemann zeta-function. *Lith. Math. J.*, **35**(4):399–402, 1995.

[9] A. Laurinčikas. On the Matsumoto zeta-function. *Acta Arith.*, **84**(1):1–16, 1998.

[10] A. Laurinčikas. Joint universality of zeta-functions with periodic coefficients. *Izv. Math.*, **74**(3):515–539, 2010.

[11] A. Laurinčikas. Universality of composite functions of periodic zeta-functions. *Math. Sb.*, **203**(11):1631–1646, 2012. https://doi.org/10.1070/SM2012v203n11ABEH004279.

[12] A. Laurinčikas. The joint discrete universality of periodic zeta-functions. In J. Sander, J. Steuding and R. Steuding(Eds.), *From Arithmetic to Zeta-Functions*, Number Theory in Memory of Wolfgang Schwarz, pp. 231–246. Springer, 2016.

[13] A. Laurinčikas. Universality theorems for zeta-functions with periodic coefficients. *Siber. Math J.*, **57**(2):330–339, 2016. https://doi.org/10.17377/smzh.2016.57.215.

[14] A. Laurinčikas and R. Macaitienė. On the joint universality of periodic zeta-functions. *Math. Notes*, **85**(1-2):51–60, 2009. https://doi.org/10.1134/S0001434609010052.

[15] A. Laurinčikas, R. Macaitienė and D. Šiaučiūnas. The joint universality for periodic zeta-functions. *Chebysh. Sb.*, **8**(2):162–174, 2007.

[16] A. Laurinčikas and D. Šiaučiūnas. Remarks on the universality of the periodic zeta-function. *Math. Notes*, **80**(3-4):532–538, 2006. https://doi.org/10.4213/mzm2848.

[17] R. Macaitienė, M. Stoncelis and D. Šiaučiūnas. A weighted discrete universality theorem for periodic zeta-functions. In A. Dubickas et al.(Ed.), *Anal. Probab. Methods Number Theory*, Proc. of 6th Palanga Conference, pp. 97–107, Vilnius University, 2017.

[18] R. Macaitienė, M. Stoncelis and D. Šiaučiūnas. A weighted universality theorem for periodic zeta-functions. *Math. Modell. Analysis*, **22**(1):95–105, 2017. https://doi.org/10.3846/13926292.2017.1269373.

[19] S.N. Mergelyan. Uniform approximation to functions of complex variable. *Usp. Mat. Nauk*, **7**:31–122, 1952 (in Russian).

[20] H.L. Montgomery. *Topics in Multiplicative Number Theory.* Lecture Notes Math. Vol. 227, Springer-Verlag, Berlin, 1971. https://doi.org/10.1007/BFb0060851.

[21] J. Sander and J. Steuding. Joint universality for sums and products of Dirichlet L-functions. *Analysis*, **26**(3):295–312, 2006. https://doi.org/10.1524/anly.2006.26.99.295.

[22] J. Steuding. *Value-Distribution of L-Functions.* Lecture Notes Math. Vol. 1877, Springer-Verlag, Berlin, Heidelberg, 2007. https://doi.org/10.1007/978-3-540-44822-8.

[23] M. Stoncelis and D. Šiaučiūnas. On the periodic zeta-function. *Chebyshevskii Sb.*, **15**(4):139–147, 2014.

[24] S.M. Voronin. Theorem on the "universality" of the Riemann zeta-function. *Math. USSR Izv.*, **39**(3):475–486, 1975. https://doi.org/10.1070/IM1975v009n03ABEH001485.