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COMPUTING THE STABILITY BOUNDARIES FOR THE LAGRANGE TRIANGULAR SOLUTIONS IN THE ELLIPTIC RESTRICTED THREE-BODY PROBLEM

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Abstract. An algorithm is proposed for analytical computing the stability boundaries of the Lagrange triangular solutions in the elliptic restricted three-body problem. It is based on the infinite determinant method. The algorithm has been implemented by using the computer algebra system *Mathematica* and the stability boundaries have been determined in the form of power series with respect to a small parameter with accuracy up to the 10th order.

Key words: restricted three-body problem, stability, infinite determinant method

1. Introduction

The restricted problem of three bodies is a famous model of classical mechanics [9]. It was proposed by Euler more than 200 years ago but its general solution has not been found yet. Remind that the problem describes the motion of particle P_0 having infinitesimal mass in the gravitational field generated by two particles P_1 and P_2 having finite masses m_1 and m_2, respectively. These two primary particles move about their common center of mass in the orbits determined by the corresponding two-body problem.

In 1772, Lagrange proved that the particle P_0 can move in the orbital plane of the primaries in such a way that the particles are in the vertices of an equilateral triangle at any instant of time. The solutions of this kind are known as Lagrange's triangular solutions [8, 9]. The study of their stability turned out to be a very complicated problem and it is solved completely only in the case of circular orbits of the particles [4].

Studying the stability of Lagrange's triangular solutions in the elliptic case, we have to determine first the domains of their linear stability in the parameter space. The system is characterized by two parameters, namely, an eccentricity of the particles orbits e and a ratio of their masses $\mu = \dfrac{m_2}{m_1 + m_2}$. So, the problem is to find the curves $\mu = \mu(e)$ in the (μ, e) plane which are the boundaries between the domains of stability and instability of the system.

The first work in this field was done by Danby [2], who investigated the characteristic equation of the system numerically and constructed the domains of its linear stability in the (μ, e) plane. One of the most exact analytical calculations of this kind was done in [6], where perturbation methods were used and the fourth order analytical expressions for the stability boundaries were given. Recently Markeev [5] proposed an algorithm for determination of the stability boundaries for the Hamiltonian system depending on a small parameter. It is based on the canonical transformations reducing the Hamiltonian function to the normal form. Using this algorithm, he improved the calculations of Nayfeh and Kamel [6] and obtained the first four coefficients in series expansion of one stability boundary in powers of e.

The main aim of the present paper is to develop an algorithm for analytical computing the stability boundaries of Lagrange's triangular solutions using the infinite determinant method [3]. This method seems to be the most effective in computing the stability boundaries for the Hamiltonian systems with periodic coefficients [7]. The proposed algorithm is easily implemented with some modern computer algebra system such as *Mathematica* [10]. We have done the corresponding calculations and obtained the stability boundaries $\mu = \mu(e)$ in the form of power series in e with accuracy $O(e^{10})$.

2. Analysis of the Linearized Equations of the Perturbed Motion

Linearized differential equations of the perturbed motion in the neighbourhood of Lagrange's triangular solutions of the elliptic three-body problem can be written in the Hamiltonian form

$$\frac{dq_j}{d\nu} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{d\nu} = -\frac{\partial H}{\partial q_j}, \quad (j = 1, 2), \tag{2.1}$$

where the Hamiltonian function H is given by (see, for instance, [4], Chapter 7, p.125 and quoted references)

$$H = \frac{1}{2}(p_1^2 + p_2^2) + p_1 q_2 - p_2 q_1 + \frac{1 + 4e\cos\nu}{8(1 + e\cos\nu)}\, q_1^2 - \frac{\kappa}{1 + e\cos\nu}\, q_1 q_2 + \frac{-5 + 4e\cos\nu}{8(1 + e\cos\nu)}\, q_2^2 \ . \tag{2.2}$$

Here a true anomaly ν is used as an independent variable and parameter κ is defined as $\kappa = \dfrac{3\sqrt{3}(1-2\mu)}{4}$. Substituting (2.2) into (2.1), we obtain a system of linear differential equations with periodic coefficients of period $T = 2\pi$

$$\begin{cases} \dfrac{dq_1}{d\nu} = p_1 + q_2, \quad \dfrac{dq_2}{dt\nu} = p_2 - q_1, \\ \dfrac{dp_1}{d\nu} = p_2 - \dfrac{1+4e\cos\nu}{4(1+e\cos\nu)}\, q_1 + \dfrac{\kappa}{1+e\cos\nu}\, q_2, \\ \dfrac{dp_2}{d\nu} = -p_1 + \dfrac{\kappa}{1+e\cos\nu}\, q_1 - \dfrac{-5+4e\cos\nu}{4(1+e\cos\nu)}\, q_2. \end{cases} \tag{2.3}$$

Obviously, the right-hand sides of system (2.3) are analytic functions of e in the neighbourhood of the point $e = 0$ and, hence, its characteristic exponents are continuous functions of e (see, for example, [11]). Then the properties of its solutions for sufficiently small values of $e > 0$ are determined by its characteristic exponents calculated for $e = 0$. In this case system (2.3) reduces to the system of four differential equations with constant coefficients whose characteristic exponents λ are easily found and may be written as

$$\lambda_{1,2} = \pm i\sigma_1, \quad \lambda_{3,4} = \pm i\sigma_2, \tag{2.4}$$

where i is the imaginary unit ($i^2 = -1$) and

$$\sigma_{1,2} = \frac{1}{\sqrt{2}}\left(1 \pm \sqrt{1-27\mu+27\mu^2}\right)^{1/2} .$$

It can be readily seen that characteristic exponents (2.4) are different purely imaginary numbers only if the following inequality is fulfilled

$$0 < 27\mu(1-\mu) < 1 \quad \text{or} \quad 0 < \mu < \mu_* \tag{2.5}$$

where $\mu_* = \dfrac{1}{18}\,(9-\sqrt{69})$ is the Routh critical mass ratio. Thus, inequality (2.5) is a necessary condition for stability of Lagrange's triangular solutions in the case of $e = 0$.

According to the general theory of differential equations with periodic coefficients [11], system (2.3) may become unstable for sufficiently small $e > 0$ only if there exist such values of parameter μ in the interval (2.5) that the following equality is fulfilled

$$\lambda_j \pm \lambda_k = iN, \quad j,k = 1,2,3,4; \;\; N = 0, \pm 1, \pm 2, \ldots \tag{2.6}$$

Analyzing characteristic exponents (2.4), one can easily show that there is only one such value of the parameter μ, namely,

$$\mu_0 = \frac{1}{6}(3-2\sqrt{2}), \tag{2.7}$$

when $\lambda_3 = i\sigma_2$, $\lambda_4 = -i\sigma_2$, $N = 1$ or $2\sigma_2 = 1$. Hence, the domain of instability of system (2.3) can exist only in the neighbourhood of the point $P(\mu_0, 0)$ in the (μ, e) plane. The boundaries of this domain are some curves $\mu = \mu(e)$ crossing the $e = 0$ axis at point (2.7). For sufficiently small values of $e > 0$ we can represent these curves in the form of power series

$$\mu = \mu_0 + \mu_1 e + \mu_2 e^2 + \dots . \tag{2.8}$$

Now the problem is to calculate coefficients μ_k in the expansion (2.8).

3. Infinite Determinant Method

In order to simplify the calculations, we rewrite system (2.3) in the form of two second-order differential equations

$$\begin{cases} (1 + e\cos\nu)\Big(\dfrac{d^2 q_1}{d\nu^2} - 2\dfrac{dq_2}{d\nu}\Big) = \dfrac{3}{4}\, q_1 + \kappa q_2, \\ (1 + e\cos\nu)\Big(\dfrac{d^2 q_2}{d\nu^2} + 2\dfrac{dq_1}{d\nu}\Big) = \kappa q_1 + \dfrac{9}{4} q_2. \end{cases} \tag{3.1}$$

According to the Floquet–Liapunov theory (see, for example, [1]), a solution of system (3.1) can be represented in the following form

$$q_1 = \exp(i\sigma\nu)\Big(a_0 + \sum_{k=1}^{\infty} (a_k \ \cos(k\nu) + b_k \ \sin(k\nu))\Big),$$

$$q_2 = \exp(i\sigma\nu)\Big(c_0 + \sum_{k=1}^{\infty} (c_k \ \cos(k\nu) + d_k \ \sin(k\nu))\Big). \tag{3.2}$$

Substituting solution (3.2) into (3.1) and canceling the multipliers $\exp(i\sigma\nu)$ in the left- and right–hand sides of each equation, we obtain two Fourier series which must vanish for all ν. Hence, coefficients of $\cos(k\nu)$ and $\sin(k\nu)$ in each equation much vanish and we obtain an infinite sequence of equations determining the coefficients a_k, b_k, c_k, d_k in (3.2).

The first two equations in the sequence arise when we equate to zero the constant terms in the corresponding Fourier series. They are given by

$$\Big(\frac{3}{4} + \sigma^2\Big)a_0 + (\kappa + 2i\sigma)c_0 + \frac{e}{2}(1 + \sigma^2)a_1 - ie\sigma b_1 + ie\sigma c_1 + ed_1 = 0,$$

$$(\kappa - 2i\sigma)a_0 + \Big(\frac{9}{4} + \sigma^2\Big)c_0 - ie\sigma a_1 - eb_1 + \frac{e}{2}(1 + \sigma^2)c_1 - ie\sigma d_1 = 0. \tag{3.3}$$

The following four equations correspond to the coefficients of $\cos\nu$ and $\sin\nu$.

$$e\sigma^2 a_0 + 2ie\sigma c_0 + \Big(\frac{7}{4} + \sigma^2\Big)a_1 - 2i\sigma b_1 + (\kappa + 2i\sigma)c_1 + 2d_1 + \frac{e}{2}(4 + \sigma^2)a_2$$
$$- 2ie\sigma b_2 + ie\sigma c_2 + 2ed_2 = 0,$$

$$-2ie\sigma a_0 + e\sigma^2 c_0 + (\kappa - 2i\sigma)a_1 - 2b_1 + \Big(\frac{13}{4} + \sigma^2\Big)c_1 - 2i\sigma d_1 - ie\sigma a_2$$
$$- 2eb_2 + \frac{e}{2}(4 + \sigma^2)c_2 - 2ie\sigma d_2 = 0,$$

$$2i\sigma a_1 + \Big(\frac{7}{4} + \sigma^2\Big)b_1 - 2c_1 + (\kappa + 2i\sigma)d_1 + 2ie\sigma a_2 + \frac{e}{2}(4 + \sigma^2)b_2 - 2ec_2 + ie\sigma d_2 = 0,$$

$$2a_1 + (\kappa - 2i\sigma)b_1 + 2i\sigma c_1 + \Big(\frac{13}{4} + \sigma^2\Big)d_1 + 2ea_2 - ie\sigma b_2 + 2ie\sigma c_2 + \frac{e}{2}(4 + \sigma^2)d_2 = 0. \tag{3.4}$$

The rest equations are obtained by means of equating to zero coefficients of $\cos(k\nu)$ and $\sin(k\nu)$ in both equations (3.1). For any $k > 1$ we have four equivalent equations which can be written in general form as

$$\frac{e}{2}\big((k-1)^2 + \sigma^2\big)a_{k-1} - ie(k-1)\sigma b_{k-1} + ie\sigma c_{k-1} + e(k-1)d_{k-1} + \Big(\frac{3}{4} + k^2 + \sigma^2\Big)a_k - 2ik\sigma b_k + (\kappa + 2i\sigma)c_k + 2kd_k + \frac{e}{2}\big((k+1)^2 + \sigma^2\big)a_{k+1} - ie(k+1)\sigma b_{k+1} + ie\sigma c_{k+1} + e(k+1)d_{k+1} = 0,$$

$$-ie\sigma a_{k-1} - e(k-1)b_{k-1} + \frac{e}{2}((k-1)^2 + \sigma^2)c_{k-1} - ie(k-1)\sigma d_{k-1} + (\kappa - 2i\sigma)a_k - 2kb_k + \Big(\frac{9}{4} + k^2 + \sigma^2\Big)c_k - 2ik\sigma d_k - ie\sigma a_{k+1} - e(k+1)b_{k+1} + \frac{e}{2}((k+1)^2 + \sigma^2)c_{k+1} - ie(k+1)\sigma d_{k+1} = 0,$$

$$ie\sigma(k-1)a_{k-1} + \frac{e}{2}((k-1)^2 + \sigma^2)b_{k-1} - e(k-1)c_{k-1} + ie\sigma d_{k-1} + 2ik\sigma a_k + \Big(\frac{3}{4} + k^2 + \sigma^2\Big)b_k - 2kc_k + (\kappa + 2i\sigma)d_k + ie(k+1)\sigma a_{k+1} + \frac{e}{2}\big((k+1)^2 + \sigma^2\big)b_{k+1} - e(k+1)c_{k+1} + ie\sigma d_{k+1} = 0,$$

$$e(k-1)a_{k-1} - ie\sigma b_{k-1} + ie\sigma(k-1)c_{k-1} + \frac{e}{2}((k-1)^2 + \sigma^2)d_{k-1} + 2ka_k + (\kappa - 2i\sigma)b_k + 2ik\sigma c_k + \Big(\frac{9}{4} + k^2 + \sigma^2\Big)d_k + e(k+1)a_{k+1} - ie\sigma b_{k+1} + ie(k+1)\sigma c_{k+1} + \frac{e}{2}((k+1)^2 + \sigma^2)d_{k+1} = 0. \tag{3.5}$$

The infinite sequence of equations (3.3)–(3.5) determines the coefficients a_0, c_0, a_k, b_k, c_k, d_k $(k \geq 1)$ in (3.2) and is just a homogeneous system of linear algebraic equations. For a solution of (3.3)–(3.5) to exist, a determinant of the corresponding infinite matrix must vanish, thus determining some curves $\mu = \mu(e)$. Of course, it is impossible to calculate a determinant of the infinite matrix. So we should truncate the infinite sequence of equations (3.3)–(3.5) after the sth term, where s is a suitably large number. For example, taking into account the first six equations (3.3)–(3.4), we obtain the corresponding determinant in the form

$$D_1 = \begin{vmatrix} \frac{3}{4}+\sigma^2 & \kappa+2i\sigma & \frac{e}{2}(1+\sigma^2) & -ie\sigma & ie\sigma & e \\ \kappa-2i\sigma & \frac{9}{4}+\sigma^2 & -ie\sigma & -e & \frac{e}{2}(1+\sigma^2) & -ie\sigma \\ e\sigma^2 & 2ie\sigma & \frac{7}{4}+\sigma^2 & -2i\sigma & \kappa+2i\sigma & 2 \\ -2ie\sigma & e\sigma^2 & \kappa-2i\sigma & -2 & \frac{13}{4}+\sigma^2 & -2i\sigma \\ 0 & 0 & 2i\sigma & \frac{7}{4}+\sigma^2 & -2 & \kappa+2i\sigma \\ 0 & 0 & 2 & \kappa-2i\sigma & 2i\sigma & \frac{13}{4}+\sigma^2 \end{vmatrix} . \tag{3.6}$$

As for every $k \geq 1$ system (3.5) contains four equations of equivalent form, the notation D_s corresponds to the determinant of the $(4s+2)$th order matrix.

Let us remind now that κ in (3.6) is determined by the parameter μ which is represented in the form of power series (2.8). Hence, for any s the determinant D_s may be represented in the form of series expansion in powers of e as well. As it must vanish for all e, we can equate the corresponding coefficients of e^k $(k \geq 0)$ to zero. Thus, we obtain a system of algebraic equations with respect to σ and coefficients μ_1, μ_2, ... Solving this system, we can easily find σ in the form of power series in e in the neighbourhood of the point (2.7). Then the stability boundaries are determined from the condition that σ takes only real values.

4. Determination of the Stability Boundaries

In the case of $e = 0$ and $\mu = \mu_0$ we have

$$\sigma_1 = \frac{\sqrt{3}}{2}, \quad \sigma_2 = \frac{1}{2} .$$

General analysis shows (see [7, 11]) that for sufficiently small $e > 0$ only σ_2 can have an imaginary part what means an instability of the system. So, in order to find the boundaries between the domains of stability and instability in the neighbourhood of the point (2.7), let us represent σ in the form

$$\sigma = \frac{1}{2} + \sigma_{21} e + \sigma_{22} e^2 + \ldots . \tag{4.1}$$

Now we can substitute (2.8) and (4.1) into the expression for determinant D_s and expand it in powers of e. Note that coefficients of e^0 and e turn out to be equal to zero. Equating the coefficients of e^k $(k \geq 2)$ to zero, we obtain a sequence of equations, determining coefficients σ_{2j} $(j = 1, 2, \ldots)$ in the expansion (4.1). Except for the first one, they are quite cumbersome and we do not write them here. The coefficient of e^2 gives an equation

$$64\sigma_{21}^2 - 10368\mu_1^2 + 33 = 0 . \tag{4.2}$$

Obviously, σ_{21} will be real only if $|\mu_1| \geq \frac{\sqrt{66}}{144}$. Thus, the boundary values of μ_1 are given by

$$\mu_1 = \pm\frac{\sqrt{66}}{144} \ . \tag{4.3}$$

The corresponding value of σ_{21} is equal to zero. Extracting coefficient of e^3 and taking into account (4.3) and $\sigma_{21} = 0$, we obtain an equation

$$2304\sqrt{2}\mu_2 - 49 = 0 \ ,$$

which gives

$$\mu_2 = \frac{49}{2304\sqrt{2}} \ . \tag{4.4}$$

Repeating these calculations, that can be done effectively with a computer, we obtain the other coefficients μ_3, μ_4, ..., determining the stability boundaries (2.8). As a result we have found two curves, crossing $e = 0$ axis in the (μ, e) plane, in the point μ_0

$$\mu = \mu_0 \mp \mu_1 e + \mu_2 e^2 \pm \mu_3 e^3 - \mu_4 e^4 \pm \mu_5 e^5 - \mu_6 e^6 \pm \mu_7 e^7 - \mu_8 e^8 \pm \mu_9 e^9 - \mu_{10} e^{10}. \tag{4.5}$$

The values of coefficients μ_k, $k = 1, 2, \ldots, 10$ are given in Table 1.

Table 1. The coefficients μ_k in the expansion (4.5).

Coefficient	Exact	Approximation
μ_1	$\frac{\sqrt{66}}{144}$	0.0564169
μ_2	$\frac{49}{2304\sqrt{2}}$	0.0150383
μ_3	$\frac{751}{4096\sqrt{66}}$	0.0225688
μ_4	$\frac{114275}{7077888\sqrt{2}}$	0.0114165
μ_5	$\frac{1951383}{46137344\sqrt{66}}$	0.00520617
μ_6	$\frac{75233555}{10871635968\sqrt{2}}$	0.0048933
μ_7	$\frac{5887298671}{259845521408\sqrt{66}}$	0.00278887
μ_8	$\frac{2474209007681}{734748645261312\sqrt{2}}$	0.00238113
μ_9	$\frac{88294500198719}{5853799906279424\sqrt{66}}$	0.00185662
μ_{10}	$\frac{28137408232597049}{124143131103351275552\sqrt{2}}$	0.00160268

Note that coefficients in (4.5) are calculated exactly. It should be emphasized also that it is sufficient to calculate the determinant D_6 in order to find them. If we consider determinants of higher order matrix we will find the higher order corrections in (2.8). The coefficients μ_j that we have already found in (4.5) will remain the same.

We can do similar calculations and find the curve crossing the axis $e = 0$ at the boundary point $\mu = \mu_*$. In this case we should use the expansions

$$\mu = \mu_* + \mu_1 e + \mu_2 e^2 + \ldots, \tag{4.6}$$

$$\sigma = \frac{1}{\sqrt{2}} + \sigma_{11} e + \sigma_{12} e^2 + \ldots. \tag{4.7}$$

Again the boundary (4.6) is determined from the condition that all coefficients σ_{1j} in (4.7) are real numbers and determinant D_s is equal to zero. In this case the calculations are much more cumbersome than we have done above and to find the first ten coefficients in (4.6) we must consider the determinant D_{10}. So the calculations can be reasonably done only with the modern computer algebra systems such as *Mathematica*. As a result we have obtained the boundary in the form

$$\mu^{(3)} = \frac{1}{18}(9 - \sqrt{69}) + \frac{2}{3\sqrt{69}}\, e^2 + \frac{239}{552\sqrt{69}}\, e^4 + \frac{8585}{50784\sqrt{69}}\, e^6 + \frac{2429947}{18688512\sqrt{69}}\, e^8 + \frac{149783831}{1719343104\sqrt{69}}\, e^{10}. \tag{4.8}$$

Replacing zero order terms in (4.6), (4.7) with $\mu = 0$ and $\sigma = 0$, respectively, we can try to find the boundary curve crossing the $e = 0$ axis in the point $\mu = 0$. We have done the corresponding calculations using determinant D_{13} and found that $\mu_j = 0$, $j = 1, 2, \ldots, 10$. Thus the axis $\mu = 0$ is a stability boundary with accuracy $o(e^{10})$.

The obtained results may be represented as the following theorem.

Theorem 1. *The domains of instability of the Lagrange triangular solutions in the elliptic restricted three-body problem are determined in the* (μ, e) *plane by the following inequalities:*

$$\mu^{(1)} < \mu < \mu^{(2)}, \quad \mu \geq \mu^{(3)},$$

where $\mu^{(1)}$, $\mu^{(2)}$ *and* $\mu^{(3)}$ *are given by* (4.5), (4.8).

The corresponding domains of instability are shown on Fig. 1 in dark colour. Note that in the points, belonging to the curve $\mu^{(3)}$, the system is unstable in linear approximation, while in the points, belonging to the curves $\mu^{(1)}$, $\mu^{(2)}$ it demonstrates stable behaviour.

5. Conclusion

In the present paper we have developed an algorithm for analytical computing the stability boundaries for the fourth order Hamiltonian system of

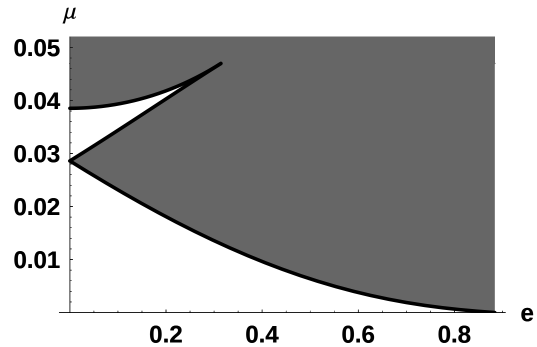

Figure 1. Domains of instability for the Lagrange triangular solutions.

linear differential equations with periodic coefficients. It is based on the infinite determinant method and can be easily implemented with the computer algebra systems, e.g. *Mathematica*. Using the algorithm, we have determined the boundaries between the domains of stability and instability in the parameter space for the Lagrange triangular solutions in the elliptic restricted three-body problem. They have been found in the form of power series in the eccentricity of the particles orbits with accuracy up to the 10th order. The obtained results are in a good agreement with similar results of [4, 6], where calculations are done with smaller accuracy and different methods are used. It should be emphasized that the proposed method is very effective not only for determining the stability boundaries but for computing characteristic exponents of the system as well.

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References

[1] L. Cesari. *Asymptotic behaviour and stability problems in ordinary differential equations.* Academic Press, New York, 1964.

[2] J.M.A. Danby. Stability of the triangular points in the elliptic restricted problem of three bodies. *Astronomical Journal*, **69**(2), 165 – 172, 1964.

[3] K.G. Lindh and P.W. Likins. Infinite determinant methods for stability analysis of periodic-coefficient differential equations. *AIAA Journal*, **8**(4), 680 – 686, 1970.

[4] A.P. Markeev. *The libration points in celestial mechanics and cosmic dynamics.* Nauka, Moscow, 1978. (in Russian)

[5] A.P. Markeev. On one special case of parametric resonance in problems of celestial mechanics. *Astronomicheskij journal, Pis'ma*, **31**(5), 388 – 394, 2005. (in Russian)

[6] A.H. Nayfeh and A.A. Kamel. Stability of the triangular points in the elliptic restricted problem of three bodies. *AIAA Journal*, **8**(2), 221 – 223, 1970.
[7] A.N. Prokopenya. Determination of the stability boundaries for the Hamiltonian systems with periodic coefficients. *Mathematical Modelling and Analysis*, **10**(2), 191 – 204, 2005.
[8] A.E. Roy. *Orbital motion.* Adam Hilger Ltd, Bristol, 1978.
[9] V. Szebehely. *Theory of orbits. The restricted problem of three bodies.* Academic Press, New York, London, 1967.
[10] S. Wolfram. *The Mathematica book.* Wolfram Media/Cambridge University Press, 1999.
[11] V.A. Yakubovich and V.M. Starzhinskii. *Linear differential equations with periodic coefficients.* John Wiley, New York, 1975.