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TEMPERATURE MODELLING WITHIN A THIN MATERIAL SHEET INVOLVED IN CONDUCTIVE-RADIATIVE HEAT TRANSFER[1]

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Abstract. In this paper we consider a problem about finding of temperature approximation within a thin material sheet involved in conductive-radiative heat transfer. As result, we found that temperature within the sheet can be approximated in L_2 norm by solution of a simple nonlinear operator equation.

Key words: elliptic equation, boundary value problem, thin layer analysis, conductive-radiative heat transfer

1. Introduction

Mathematical models for description of conductive-radiative heat transfer are considered in [5]. As a result, complicated integro-differential boundary value problems are derived that models, heat propagation within a physical system.

In [3] a mathematical model is given for the description of process of the oil burn-out from glass fabric sheets. There a glass fabric sheet is pulled with a constant speed through a furnace. As conductive–radiative heat transfer occurs in the heaters-fabric sheet system, when the glass fabric is heated up in the furnace. If temperature raises up to the critical level, oil from the fabric starts to burn out. In this paper we neglect oil burn-out process from the fabric and therefore. If we take into account only conductive-radiative heat transfer, then the resulting temperature distribution T_δ in the glass fabric sheet Ω_δ of thickness 2δ can be found after solving the following elliptic boundary value problem:

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$$k_1 \Delta T_\delta - k_2 \frac{\partial T_\delta}{\partial x_1} = 0 \text{ in } \Omega_\delta, \tag{1.1}$$

$$k_1 \frac{\partial T_\delta}{\partial \nu} = -G_\delta^1(|T_\delta|^3 T_\delta) + G_\delta^2(|T^h|^3 T^h) - k_3(T_\delta - T_\delta^g) \text{ on } \Sigma_\delta^s \subset \partial\Omega_\delta, \tag{1.2}$$

$$T_\delta = T_\delta^i \text{ on } \Sigma_\delta^i \subset \Omega_\delta \backslash \Sigma_\delta^s, \quad k_1 \frac{\partial T_\delta}{\partial \nu} = 0 \text{ on } \partial\Omega_\delta \backslash (\Sigma_\delta^s \cup \Sigma_\delta^i). \tag{1.3}$$

Here linear operators

$$G_\delta^1 : L_{5/4}(\Sigma_\delta^s) \mapsto L_{5/4}(\Sigma_\delta^s), \;\; G_\delta^2 : L_{5/4}(\Sigma^h) \mapsto L_{5/4}(\Sigma_\delta^s)$$

are given in the implicit form and they describe radiative heat propagation within Σ_δ^s, Σ^h system, where Σ_δ^s is active surface of Ω_δ and Σ^h is active surface of the furnace. Both surfaces can emit, absorb and reflect radiation. T^h is temperature distribution on Σ^h, whereas T_δ^g is temperature of the surrounding air.

Unfortunately, it is very hard to carry out any numerical calculations for the boundary value problem (1.1) – (1.3) with standard numerical methods. The main reason is the fact that the sheet geometry is strongly degenerated in this situation. Thickness-width ratio for the sheet can achieve the value $\frac{1}{15000}$.

In this paper we have analyzed the dependence of the weak solutions T_δ of the boundary value problem (1.1) – (1.3) on the thickness 2δ and found, that, as $\delta \to 0$, the weak solutions T_δ can be approximated in $L_2(\Omega_\delta)$ norm by solutions of a simple nonlinear operator equation.

2. Preliminaries

In this section we give the formulation of the full integro-differential boundary value problem and list some preliminary results. It is important to note that we widely use the methodology from paper [5], which deals with similar mathematical models of conductive-radiative heat transfer. Let

$$\Omega_\delta := (-l_1, l_1) \times (-l_2, l_2) \times (-\delta, \delta)$$

be a glass fabric sheet of thickness 2δ ($l_1 > 0$, $l_2 > 0$, $l_3 > 0$, $0 < \delta \le l_3$) pulled in x_1 axis direction through a furnace. Let us denote faces of Ω_δ by:

$$\Sigma_\delta^+ := \{x \in \partial\Omega_\delta : x_3 = \delta\}, \;\; \Sigma_\delta^- := \{x \in \partial\Omega_\delta : x_3 = -\delta\},$$
$$\Sigma_\delta^{'} := \{x \in \partial\Omega_\delta : x_2 = -l_2\}, \;\; \Sigma_\delta^{''} := \{x \in \partial\Omega_\delta : x_2 = l_2\},$$
$$\Sigma_\delta^i := \{x \in \partial\Omega_\delta : x_1 = -l_1\}, \;\; \Sigma_\delta^o := \{x \in \partial\Omega_\delta : x_1 = l_1\}.$$

We assume that only one part of $\partial\Omega_\delta$ is involved in radiative heat exchange:

$$\Sigma_\delta^s := \Sigma_\delta^- \cup \Sigma_\delta^+ \cup \Sigma_\delta^{'} \cup \Sigma_\delta^{''}.$$

Let the furnace is formed from finite set of heaters $\{\Omega_i \subset \mathbb{R}^3 : i = 1,..,n\}$. For mathematical reasons we assume that each heater is a bounded Lipschitz domain with piecewise Lyapunov boundary and it lies at a positive distance from other heaters and from Ω_δ. Let us denote the overall heater surface of the furnace by

$$\Sigma^h := \bigcup_{i\in\{1,..,n\}} \partial\Omega_i.$$

Let us also denote

$$Q := [-l_1, l_1] \times [-l_2, l_2], \quad \Sigma^r_\delta := \Sigma^s_\delta \cup \Sigma^h, \quad \Omega^r_\delta := (\bigcup_{i\in\{1,..,n\}} \Omega_i) \cup \Omega_\delta.$$

In what follows c and M with subscripts will denote nonnegative constants that do not depend on δ.

We denote by $\mathfrak{L}(X, Y)$ a space of the linear bounded operators that maps a Banach space X into a Banach space Y. Let I be an identity operator.

We denote by $L_p(\cdot)$ ithe standard Lebesgue spaces ($1 \le p \le \infty$) and by $W^1_2(\cdot)$, $W^1_\infty(\cdot)$ the standard Sobolev spaces. Let $V_5(\Omega_\delta)$, $\dot{V}_5(\Omega_\delta)$ be Banach spaces:

$$V_5(\Omega_\delta) := \{u \in W^1_2(\Omega_\delta) : u\,|_{\Sigma^s_\delta} \in L_5(\Sigma^s_\delta)\},$$
$$\dot{V}_5(\Omega_\delta) := \{u \in V_5(\Omega_\delta) : u\,|_{\Sigma^i_\delta} = 0\}.$$

If we make substitution $T_\delta \mapsto u_\delta + \phi_\delta$, then we can rewrite the boundary value problem (1.1) – (1.3) in the weak form:

$$\begin{aligned}\int_{\Omega_\delta} \Big(k_1\big(\nabla(u_\delta+\phi_\delta)\cdot\nabla\psi\big) + k_2(u_\delta+\phi_\delta)_{x_1}\psi\Big)\,dv \\ + \int_{\Sigma^s_\delta} G^1_\delta\big(|u_\delta+\phi_\delta|^3(u_\delta+\phi_\delta)\big)\psi\,ds + \int_{\Sigma^s_\delta} k_3(u_\delta+\phi_\delta)\psi\,ds \\ = \int_{\Sigma^s_\delta} G^2_\delta(|T^h|^3 T^h)\psi\,ds + \int_{\Sigma^s_\delta} k_3 T^g_\delta \psi\,ds, \quad \forall\psi \in \dot{V}_5(\Omega_\delta),\end{aligned} \tag{2.1}$$

where $u_\delta \in \dot{V}_5(\Omega_\delta)$ is unknown variable and

$$k_1 > 0,\;\; k_2 > 0,\;\; k_3 > 0,\;\; \phi_\delta \in V_5(\Omega_\delta), \phi_\delta = T^i_\delta \text{ a.e. on } \Sigma^i_\delta,$$
$$T^i_\delta \in W^1_\infty(\Sigma^i_\delta), \quad 0 \le T^i_\delta \le M_1 \text{ a.e. on } \Sigma^i_\delta,$$
$$T^h \in L_\infty(\Sigma^h), \quad 0 \le T^h \le M_1 \text{ a.e. on } \Sigma^h,$$
$$T^g_\delta \in L_\infty(\Sigma^s_\delta), \quad 0 \le T^g_\delta \le M_1 \text{ a.e. on } \Sigma^s_\delta,$$
$$G^1_\delta \in \mathfrak{L}(L_{5/4}(\Sigma^s_\delta), L_{5/4}(\Sigma^s_\delta)),\;\; G^2_\delta \in \mathfrak{L}(L_{5/4}(\Sigma^h), L_{5/4}(\Sigma^s_\delta)).$$

Further we assume that $T^g_\delta\,|_{\Sigma^+_\delta}$, $T^g_\delta\,|_{\Sigma^-_\delta}$ do not depend on δ. We also assume that $T^g_\delta\,|_{\Sigma'_\delta}$, $T^g_\delta\,|_{\Sigma''_\delta}$ depend only on variable x_1, T^i_δ depend only on variable x_2 and there exists a constant M_2 such that

$$(T^g_\delta\,|_\Sigma) \in W^1_\infty(\Sigma), \quad \|(T^g_\delta\,|_\Sigma)\|_{W^1_\infty(\Sigma)} \le M_2,$$

if Σ is one of the surfaces Σ^+_δ, Σ^-_δ, Σ'_δ and Σ''_δ.

Originally mathematical formalization of conductive-radiative heat exchange process in the furnace-fabric sheet system gives us the following system:

$$\int_{\Omega_\delta}\Big(k_1\big(\nabla(u_\delta+\phi_\delta)\cdot\nabla\psi\big)+k_2(u_\delta+\phi_\delta)_{x_1}\psi\Big)\,dv+\int_{\Sigma^s_\delta}k_3(u_\delta+\phi_\delta)\psi\,ds$$
$$+\int_{\Sigma^s_\delta}g_\delta\psi\,ds=\int_{\Sigma^s_\delta}k_3T^g_\delta\psi\,ds\quad\forall\psi\in\dot V_5(\Omega_\delta), \tag{2.2}$$

$$\rho_\delta(x)-\big(1-\epsilon_\delta(x)\big)\int_{\Sigma^r_\delta}k_\delta(x,y)\rho_\delta(y)\,ds(y)$$
$$=\epsilon_\delta(x)\sigma|q_\delta(x)|^3q_\delta(x)\text{ a.e. }x\in\Sigma^r_\delta, \tag{2.3}$$

$$g_\delta(x)=\rho_\delta(x)-\int_{\Sigma^r_\delta}k_\delta(x,y)\rho_\delta(y)\,ds(y)\text{ a.e. }x\in\Sigma^s_\delta, \tag{2.4}$$

$$q_\delta\mid_{\Sigma^s_\delta}=u_\delta+\phi_\delta,\quad q_\delta\mid_{\Sigma^h}=T^h, \tag{2.5}$$

where

$$\sigma>0,\ \ \epsilon_0>0,\ \ \epsilon_\delta\in L_\infty(\Sigma^r_\delta),\ \ \epsilon_0\le\epsilon_\delta\le1\text{ a.e. on }\Sigma^r_\delta.$$

However, under some assumptions this system can be transformed into the boundary value problem (2.1) ((1.1) – (1.3)).

Further we assume that the functions $\epsilon_\delta\mid_{\Sigma^+_\delta}$, $\epsilon_\delta\mid_{\Sigma^-_\delta}$ do not depend on δ and let us denote , them as ϵ^+ and ϵ^- respectively. We also assume that $\epsilon_\delta\mid_{\Sigma'_\delta}$, $\epsilon_\delta\mid_{\Sigma''_\delta}$ depend only on variable x_1 and there exits a constant M_3 such that

$$(\epsilon_\delta\mid_\Sigma)\in W^1_\infty(\Sigma),\quad\|(\epsilon_\delta\mid_\Sigma)\|_{W^1_\infty(\Sigma)}\le M_3,$$

if Σ is one of the surfaces Σ^+_δ, Σ^-_δ, $\Sigma^{'}_\delta$ and $\Sigma^{''}_\delta$.

The kernel $k_\delta:\Sigma^r_\delta\times\Sigma^r_\delta\mapsto\mathbb{R}$ from (2.3), (2.4) is defined as

$$k_\delta(x,y):=w_\delta(x,y)\theta_\delta(x,y),$$
$$w_\delta(x,y):=\frac{\cos\left(\nu(x),(y-x)\right)\cos\left(\nu(y),(x-y)\right)}{\pi|x-y|^2},$$
$$\theta_\delta(x,y):=\begin{cases}1,&\text{if }\{z\in\mathbb{R}^3:z=\tau x+(1-\tau)y,0<\tau<1\}\cap\Omega^r_\delta=\emptyset,\\0,&\text{otherwise,}\end{cases}$$

where $\nu(\cdot)$ denotes the outward normal of the surface Σ^r_δ ($\nu(\cdot)$ exists almost everywhere on Σ^r_δ, as it is Lipschitz surface).

The analytical properties of the function $k_\delta(x,y)$ and the fact that Σ^r_δ is Lipschitz surface allow us by involving the Gauss's formula to get the following estimate:

$$0\le\int_{\Sigma^r_\delta}k_\delta(x,y)\,ds(y)=\lim_{\tau\to0}\int_{\Sigma^r_\delta\setminus B(x,\tau)}k_\delta(x,y)\,ds(y)$$
$$=\lim_{\tau\to0}\int_{S^+(x,\tau)\cap Q(x,\tau)}\frac{\cos\left(\nu(x),(y-x)\right)}{\pi\tau^2}\,ds(y)\le1\text{ a.e. }x\in\Sigma^r_\delta, \tag{2.6}$$

where

$$B(x,\tau) := \{z \in \mathbb{R}^3 : \ \ |z-x| \le \tau\},$$
$$S^+(x,\tau) := \{z \in \mathbb{R}^3 : \ \ |z-x| = \tau, \ (\nu(x)\cdot(z-x)) \ge 0\},$$
$$Q(x,\tau) := \{z \in \mathbb{R}^3 : \ \ z = x + ty, \ y \in \Sigma_\delta^r \backslash B(x,\tau), \ t \ge 0\}.$$

Cylindrical shape of the domain Ω_δ and positive distance between the surfaces Σ_δ^s, Σ^h guaranty that there exists a constant $0 \le c_1 < 1$ such that for every $\tau > 0$ and for a.e. $x \in \Sigma_\delta^s$ the following estimate holds:

$$\text{meas}(S^+(x,\tau) \cap Q(x,\tau)) \le 2c_1\tau^2.$$

That yields the existence of a much stronger local estimate on the surface Σ_δ^s than (2.6). Respectively, there exists a constant $c_2 < 1$ such that:

$$0 \le \int_{\Sigma_\delta^r} k_\delta(x,y)\, ds(y) \le c_2 \text{ a.e. } x \in \Sigma_\delta^s. \tag{2.7}$$

The estimate (2.6) implies that the operator $K_\delta(u) := \int_{\Sigma_\delta^r} k_\delta(x,y)u(y)\, ds(y)$ maps $L_p(\Sigma_\delta^r)$ into $L_p(\Sigma_\delta^r)$, $K_\delta \in \mathfrak{L}(L_p(\Sigma_\delta^r), L_p(\Sigma_\delta^r))$ and

$$\|K_\delta\|_{\mathfrak{L}(L_p(\Sigma_\delta^r), L_p(\Sigma_\delta^r))} \le 1 \tag{2.8}$$

for every $1 \le p \le \infty$ ([4]). For a fixed measurable set $\Sigma \subset \Sigma_\delta^r$ we define operators

$$P_\delta^1(\Sigma) \in \mathfrak{L}(L_{5/4}(\Sigma), L_{5/4}(\Sigma_\delta^r)), \ \ P_\delta^2(\Sigma) \in \mathfrak{L}(L_{5/4}(\Sigma_\delta^r), L_{5/4}(\Sigma)),$$
$$E_\delta^1 \in \mathfrak{L}(L_{5/4}(\Sigma_\delta^r), L_{5/4}(\Sigma_\delta^r)), \ \ E_\delta^2(\Sigma) \in \mathfrak{L}(L_{5/4}(\Sigma), L_{5/4}(\Sigma_\delta^r)),$$
$$F_\delta \in \mathfrak{L}(L_{5/4}(\Sigma_\delta^r), L_{5/4}(\Sigma_\delta^r)),$$

where

$$P_\delta^1(\Sigma)(u) := \begin{cases} u(x) & x \in \Sigma, \\ 0 & x \in \Sigma_\delta^r \backslash \Sigma, \end{cases} \qquad P_\delta^2(\Sigma)(u) := u\,|_\Sigma,$$
$$E_\delta^1(u) := (1-\epsilon_\delta)u, \qquad E_\delta^2(\Sigma)(u) := P_\delta^1(\Sigma)(\sigma\epsilon_\delta\,|_\Sigma\, u),$$
$$F_\delta(u) := E_\delta^2(\Sigma_\delta^r)K_\delta\Big(\sum_{i=0}^{\infty}(E_\delta^1 K_\delta)^i\Big)(u).$$

As estimate (2.8) holds, then it allows us to exclude the variables g_δ, ρ_δ, q_δ from system (2.2)–(2.5) and to obtain the integral equality (2.1). Then operators G_δ^1, G_δ^2 from (2.1) will have the following form ([5]):

$$G_\delta^1 = P_\delta^2(\Sigma_\delta^s)(I - F_\delta)E_\delta^2(\Sigma_\delta^s), \quad G_\delta^2 = P_\delta^2(\Sigma_\delta^s)F_\delta E_\delta^2(\Sigma^h)$$

and for $\Sigma_1, \Sigma_2 \subset \Sigma_\delta^s$, then the following will hold ([5]):

$$P_\delta^2(\Sigma_2)F_\delta E_\delta^2(\Sigma_1) \in \mathfrak{L}(L_p(\Sigma_1), L_p(\Sigma_2)) \text{ for all } 5/4 \le p \le \infty,$$

$P^2_\delta(\Sigma_2)F_\delta E^2_\delta(\Sigma^h) \in \mathfrak{L}(L_p(\Sigma^h), L_p(\Sigma_2))$ for all $5/4 \le p \le \infty$,
$P^2_\delta(\Sigma_2)F_\delta P^1_\delta(\Sigma_1) \in \mathfrak{L}(L_p(\Sigma_1), L_p(\Sigma_2))$ for all $5/4 \le p \le \infty$,
$\|P^2_\delta(\Sigma_2)F_\delta P^2_\delta(\Sigma_1)\|_{\mathfrak{L}(L_p(\Sigma_1),L_p(\Sigma_2))} \le c_3 < 1$ for all $5/4 \le p \le \infty$,
if $u \in L_{5/4}(\Sigma^h)$ and $u \ge 0$ a.e. on Σ^h, then $P^2_\delta(\Sigma_2)F_\delta E^2_\delta(\Sigma^h)(u) \ge 0$ a.e. on Σ_2,
if $u \in L_{5/4}(\Sigma_1)$ and $u \ge 0$ a.e. on Σ_1, then $P^2_\delta(\Sigma_2)F_\delta E^2_\delta(\Sigma_1)(u) \ge 0$ a.e. on Σ_2.

Moreover, one can show that, if $u_1 \in L_\infty(\Sigma)$, $\|u_1\|_{L_\infty(\Sigma)} \le M_1$, $u_2 \in L_\infty(\Sigma^s_\delta)$, $\|u_2\|_{L_\infty(\Sigma^s_\delta)} \le M_1$, $u_3 \in L_\infty(\Sigma^h)$, $\|u_3\|_{L_\infty(\Sigma^h)} \le M_1$ and Σ is one of the Σ^+_δ, Σ^-_δ, $\Sigma^{'}_\delta$, $\Sigma^{''}_\delta$, then there exists a constant $M_4 \ge 0$, such that

$$P^2_\delta(\Sigma)F_\delta E^2_\delta(\Sigma)(u_1) \in W^1_\infty(\Sigma), \;\; \|P^2_\delta(\Sigma)F_\delta E^2_\delta(\Sigma)(u_1)\|_{W^1_\infty(\Sigma)} \le M_4,$$
$$P^2_\delta(\Sigma)F_\delta E^2_\delta(\Sigma^s_\delta)(u_2) \in W^1_\infty(\Sigma), \;\; \|P^2_\delta(\Sigma)F_\delta E^2_\delta(\Sigma^s_\delta)(u_2)\|_{W^1_\infty(\Sigma)} \le M_4,$$
$$P^2_\delta(\Sigma)F_\delta E^2_\delta(\Sigma^h)(u_3) \in W^1_\infty(\Sigma), \;\; \|P^2_\delta(\Sigma)F_\delta E^2_\delta(\Sigma^h)(u_3)\|_{W^1_\infty(\Sigma)} \le M_4.$$

If we adapt results from [1], [2], then we can get existence and uniform boundedness of solutions for the boundary value problem (2.1). As the form

$$\big((u_\delta, \psi) \in \dot{V}_5(\Omega_\delta) \times \dot{V}_5(\Omega_\delta)\big) \mapsto$$
$$\int_{\Omega_\delta} (k_1(\nabla(u_\delta + \phi_\delta) \cdot \nabla\psi) + k_2(u_\delta + \phi_\delta)_{x_1}\psi)\, dv + \int_{\Sigma^s_\delta} k_3(u_\delta + \phi_\delta)\psi\, ds$$

defines a monotone operator $Q_1 : \dot{V}_5(\Omega_\delta) \mapsto (\dot{V}_5(\Omega_\delta))^*$, the form

$$\big((u_\delta, \psi) \in \dot{V}_5(\Omega_\delta) \times \dot{V}_5(\Omega_\delta)\big) \mapsto \int_{\Sigma^s_\delta} G^1_\delta(|u_\delta + \phi_\delta|^3(u_\delta + \phi_\delta))\psi\, ds$$

defines a weakly continuous operator $Q_2 : \dot{V}_5(\Omega_\delta) \mapsto (\dot{V}_5(\Omega_\delta))^*$ and the estimate (2.7) guarantees that the operator $Q_1 + Q_2$ is coercive, then the boundary value problem (2.1) will have at least one weak solution $u_\delta \in \dot{V}_5(\Omega_\delta)$. Even more, the following estimate will hold:

$$0 \le (u_\delta + \phi_\delta) \le M_1 \text{ a.e. on } \Omega_\delta. \tag{2.9}$$

Let:

$$h^+_\delta := (G^2_\delta(|T^h|^3T^h) + k_3T^g_\delta + P^2_\delta(\Sigma^s_\delta)F_\delta E^2_\delta(\Sigma^s_\delta)(|u_\delta + \phi_\delta|^3(u_\delta + \phi_\delta)))\,|_{\Sigma^+_\delta},$$
$$h^-_\delta := (G^2_\delta(|T^h|^3T^h) + k_3T^g_\delta + P^2_\delta(\Sigma^s_\delta)F_\delta E^2_\delta(\Sigma^s_\delta)(|u_\delta + \phi_\delta|^3(u_\delta + \phi_\delta)))\,|_{\Sigma^-_\delta},$$
$$h^{'}_\delta := (G^2_\delta(|T^h|^3T^h) + k_3T^g_\delta + P^2_\delta(\Sigma^s_\delta)F_\delta E^2_\delta(\Sigma^s_\delta)(|u_\delta + \phi_\delta|^3(u_\delta + \phi_\delta)))\,|_{\Sigma^{'}_\delta},$$
$$h^{''}_\delta := (G^2_\delta(|T^h|^3T^h) + k_3T^g_\delta + P^2_\delta(\Sigma^s_\delta)F_\delta E^2_\delta(\Sigma^s_\delta)(|u_\delta + \phi_\delta|^3(u_\delta + \phi_\delta)))\,|_{\Sigma^{''}_\delta},$$

$$\dot{h}_\delta^+ := (G_\delta^2(|T^h|^3T^h) + k_3T_\delta^g + P_\delta^2(\Sigma_\delta^s)F_\delta E_\delta^2(\Sigma_\delta^+)(|u_\delta + \phi_\delta|^3(u_\delta + \phi_\delta))) \mid_{\Sigma_\delta^+},$$

$$\dot{h}_\delta^- := (G_\delta^2(|T^h|^3T^h) + k_3T_\delta^g + P_\delta^2(\Sigma_\delta^s)F_\delta E_\delta^2(\Sigma_\delta^-)(|u_\delta + \phi_\delta|^3(u_\delta + \phi_\delta))) \mid_{\Sigma_\delta^-},$$

$$h_\delta := G_\delta^2(|T^h|^3T^h) + k_3T_\delta^g + P_\delta^2(\Sigma_\delta^s)F_\delta E_\delta^2(\Sigma_\delta^s)(|u_\delta + \phi_\delta|^3(u_\delta + \phi_\delta)),$$

$$f_\delta^+ := (G_\delta^2(|T^h|^3T^h) + k_3T_\delta^g) \mid_{\Sigma_\delta^+}, \quad f_\delta^- := (G_\delta^2(|T^h|^3T^h) + k_3T_\delta^g) \mid_{\Sigma_\delta^-} .$$

Then (2.9) and properties of involved operators imply existence of a constant M_5 such that:

$$h_\delta^+ \in W_\infty^1(\Sigma_\delta^+), \quad h_\delta^- \in W_\infty^1(\Sigma_\delta^-), \quad h_\delta^{'} \in W_\infty^1(\Sigma_\delta^{'}), \quad h_\delta^{''} \in W_\infty^1(\Sigma_\delta^{''}),$$

$$\dot{h}_\delta^+ \in W_\infty^1(\Sigma_\delta^+), \quad \dot{h}_\delta^- \in W_\infty^1(\Sigma_\delta^-), \quad f_\delta^+ \in W_\infty^1(\Sigma_\delta^+), \quad f_\delta^- \in W_\infty^1(\Sigma_\delta^-),$$

$$h_\delta^+ \geq 0, \quad \dot{h}_\delta^+ \geq 0, \quad f_\delta^+ \geq 0 \text{ a.e. on } \Sigma_\delta^+,$$

$$h_\delta^- \geq 0, \quad \dot{h}_\delta^- \geq 0, \quad f_\delta^- \geq 0 \text{ a.e. on } \Sigma_\delta^-,$$

$$h_\delta^{'} \geq 0 \text{ a.e. on } \Sigma_\delta^{'}, \quad h_\delta^{''} \geq 0 \text{ a.e. on } \Sigma_\delta^{''},$$

$$\|h_\delta^+\|_{W_\infty^1(\Sigma_\delta^+)} \leq M_5, \quad \|\dot{h}_\delta^+\|_{W_\infty^1(\Sigma_\delta^+)} \leq M_5, \quad \|h_\delta^-\|_{W_\infty^1(\Sigma_\delta^-)} \leq M_5,$$

$$\|\dot{h}_\delta^-\|_{W_\infty^1(\Sigma_\delta^-)} \leq M_5, \quad \|h_\delta^{'}\|_{W_\infty^1(\Sigma_\delta^{'})} \leq M_5, \quad \|h_\delta^{''}\|_{W_\infty^1(\Sigma_\delta^{''})} \leq M_5.$$

Moreover, the boundary value problem (2.1) can be rewritten in the following form:

$$\int_{\Omega_\delta} \Big(k_1\big(\nabla(u_\delta + \phi_\delta)\cdot\nabla\psi\big) + k_2(u_\delta + \phi_\delta)_{x_1}\psi\Big)\, dv + \int_{\Sigma_\delta^s} \sigma\epsilon_\delta|u_\delta + \phi_\delta|^3(u_\delta + \phi_\delta)\psi\, ds + \int_{\Sigma_\delta^s} k_3(u_\delta + \phi_\delta)\psi\, ds = \int_{\Sigma_\delta^s} h_\delta\psi\, ds \quad \forall\psi \in \dot{V}_5(\Omega_\delta). \tag{2.10}$$

3. Estimate for Gradients

Let us choose a function $\eta_\delta : \Omega_\delta \mapsto \mathbb{R}$ defined as

$$\eta_\delta(x) := \min\{\frac{\text{dist}\{\Sigma_\delta^i, x\}l_3^r}{l_1\delta^r}, 1\}$$

for some fixed $0 < r < 1$. Then it is easy to verify that

$$\int_{\Omega_\delta} |\nabla\eta_\delta|^2\, dv \leq \frac{4l_3^r l_2}{l_1}\delta^{1-r}. \tag{3.1}$$

If we define a function $\psi_\delta := (u_\delta + \phi_\delta)\eta_\delta^2$, then $\psi_\delta \in \dot{V}_5(\Omega_\delta)$.

As functions $u_\delta + \phi_\delta$, h_δ, ψ_δ are uniformly bounded with respect to δ, then

$$
\begin{aligned}
&|\int_{\Sigma_\delta^s} \sigma\epsilon_\delta |u_\delta + \phi_\delta|^3 (u_\delta + \phi_\delta)\psi_\delta \, ds| \\
&\qquad + |\int_{\Sigma_\delta^s} k_3 (u_\delta + \phi_\delta)\psi_\delta \, ds| + |\int_{\Sigma_\delta^s} h_\delta \psi_\delta \, ds| \le c_4. \quad (3.2)
\end{aligned}
$$

For every chosen $\tau > 0$ it follows that

$$
\begin{aligned}
|\int_{\Omega_\delta} k_2 (u_\delta + \phi_\delta)_{x_1} \psi_\delta \, dv| &= |\int_{\Omega_\delta} k_2 (u_\delta + \phi_\delta)_{x_1} (u_\delta + \phi_\delta)\eta_\delta^2 \, dv| \\
&\le k_2 \tau \int_{\Omega_\delta} (u_\delta + \phi_\delta)_{x_1}^2 \eta_\delta^2 \, dv + \frac{k_2}{4\tau} \int_{\Omega_\delta} (u_\delta + \phi_\delta)^2 \eta_\delta^2 \, dv
\end{aligned}
$$

and, therefore, if we put $\tau = \dfrac{k_1}{4k_2}$, then, after taking into account uniform boundedness of $u_\delta + \phi_\delta$ with respect to δ, we obtain

$$
|\int_{\Omega_\delta} k_2 (u_\delta + \phi_\delta)_{x_1} \psi_\delta \, dv| \le \frac{k_1}{4} \int_{\Omega_\delta} |\nabla (u_\delta + \phi_\delta)|^2 \eta_\delta^2 \, dv + c_5. \quad (3.3)
$$

Again, for every chosen $\tau > 0$ it follows that

$$
\begin{aligned}
\int_{\Omega_\delta} k_1 (\nabla (u_\delta + \phi_\delta) \cdot \nabla \psi_\delta) \, dv &\ge k_1 \int_{\Omega_\delta} |\nabla (u_\delta + \phi_\delta)|^2 \eta_\delta^2 \, dv \\
&- k_1 \tau \int_{\Omega_\delta} |\nabla (u_\delta + \phi_\delta)|^2 \eta_\delta^2 \, dv - \frac{k_1}{4\tau} \int_{\Omega_\delta} |\nabla \eta_\delta|^2 (u_\delta + \phi_\delta)^2 \, dv
\end{aligned}
$$

and, if we put $\tau = 1/4$, then after taking into account uniform boundedness of $u_\delta + \phi_\delta$ with respect to δ and (3.1), we will have

$$
\int_{\Omega_\delta} k_1 (\nabla (u_\delta + \phi_\delta) \cdot \nabla \psi_\delta) \, dv \ge \frac{3k_1}{4} \int_{\Omega_\delta} |\nabla (u_\delta + \phi_\delta)|^2 \eta_\delta^2 \, dv - c_6. \quad (3.4)
$$

Combining (2.10), (3.2) – (3.4), we finally get

$$
\int_{\Omega_\delta} |\nabla (u_\delta + \phi_\delta)|^2 \eta_\delta^2 \, dv \le c_7. \quad (3.5)
$$

4. Main Estimate

Let us choose functions $\dot{h}_\delta : \Omega_\delta \mapsto \mathbb{R}$, $\dot{\epsilon}_\delta : \Omega_\delta \mapsto \mathbb{R}$ defined as:

$$
\begin{aligned}
\dot{h}_\delta(x_1, x_2, x_3) &:= \frac{\dot{h}_\delta^+(x_1, x_2) + \dot{h}_\delta^-(x_1, x_2)}{2}, \\
\dot{\epsilon}_\delta(x_1, x_2, x_3) &:= \frac{\epsilon^+(x_1, x_2) + \epsilon^-(x_1, x_2)}{2}.
\end{aligned}
$$

Again, if we define a function

$$\psi_\delta := (\sigma\dot{\epsilon}_\delta(u_\delta+\phi_\delta)^4 + k_3(u_\delta+\phi_\delta) - \dot{h}_\delta)\eta_\delta^2,$$

then $\psi_\delta \in \dot{V}_5(\Omega_\delta)$.

As $\|\dot{\epsilon}_\delta\|_{W^1_\infty(\Omega_\delta)} \le M_3$, $\|\dot{h}_\delta\|_{W^1_\infty(\Omega_\delta)} \le M_5$ and functions $u_\delta+\phi_\delta$, h_δ are uniformly bounded with respect to δ, then, if we take into account (3.1), (3.5), then:

$$\begin{aligned}\int_{\Omega_\delta} |\nabla\psi_\delta|^2\,dv &\le \int_{\Omega_\delta} 48(\sigma\dot{\epsilon}_\delta(u_\delta+\phi_\delta)^3+k_3)^2\eta_\delta^4|\nabla(u_\delta+\phi_\delta)|^2\,dv\\ &+ \int_{\Omega_\delta} 12\sigma^2(u_\delta+\phi_\delta)^8\eta_\delta^4|\nabla\dot{\epsilon}_\delta|^2\,dv + \int_{\Omega_\delta} 12\eta_\delta^4|\nabla\dot{h}_\delta|^2\,dv\\ &+ \int_{\Omega_\delta} 24(\sigma\dot{\epsilon}_\delta(u_\delta+\phi_\delta)^4+k_3(u_\delta+\phi_\delta)-\dot{h}_\delta)^2\eta_\delta^2|\nabla\eta_\delta|^2\,dv \le c_8 \end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\int_{\Omega_\delta} k_1(\nabla(u_\delta+\phi_\delta)\cdot\nabla\psi_\delta)\,dv &= k_1\int_{\Omega_\delta}(4\sigma\dot{\epsilon}_\delta(u_\delta+\phi_\delta)^3+k_3)\eta_\delta^2|\nabla(u_\delta+\phi_\delta)|^2\,dv\\ &+ k_1\int_{\Omega_\delta}\big(\sigma(u_\delta+\phi_\delta)^4\eta_\delta^2(\nabla(u_\delta+\phi_\delta)\cdot\nabla\dot{\epsilon}_\delta) - \eta_\delta^2(\nabla(u_\delta+\phi_\delta)\cdot\nabla\dot{h}_\delta)\big)\,dv\\ &+ k_1\int_{\Omega_\delta} 2\big(\sigma\dot{\epsilon}_\delta(u_\delta+\phi_\delta)^4+k_3(u_\delta+\phi_\delta)-\dot{h}_\delta)\eta_\delta(\nabla(u_\delta+\phi_\delta)\cdot\nabla\eta_\delta\big)\,dv\\ &\ge -k_1\Big(\int_{\Omega_\delta}\sigma^2(u_\delta+\phi_\delta)^8\eta_\delta^2|\nabla\dot{\epsilon}_\delta|^2\,dv\Big)^{1/2}\Big(\int_{\Omega_\delta}|\nabla(u_\delta+\phi_\delta)|^2\eta_\delta^2\,dv\Big)^{1/2}\\ &- k_1\Big(\int_{\Omega_\delta}\eta_\delta^2|\nabla\dot{h}_\delta|^2\,dv\Big)^{1/2}\Big(\int_{\Omega_\delta}|\nabla(u_\delta+\phi_\delta)|^2\eta_\delta^2\,dv\Big)^{1/2}\\ &- k_1\Big(\int_{\Omega_\delta}(\sigma\dot{\epsilon}_\delta(u_\delta+\phi_\delta)^4+k_3(u_\delta+\phi_\delta)-\dot{h}_\delta)^2|\nabla\eta_\delta|^2\,dv\Big)^{1/2}\\ &\times\Big(\int_{\Omega_\delta}|\nabla(u_\delta+\phi_\delta)|^2\eta_\delta^2\,dv\Big)^{1/2} \ge -c_9\delta^{(1-r)/2}. \end{aligned}\tag{4.2}$$

In addition

$$\begin{aligned}&\int_{\Omega_\delta} k_2(u_\delta+\phi_\delta)_{x_1}\psi_\delta\,dv\\ &\ge -k_2\int_{\Omega_\delta}|(u_\delta+\phi_\delta)_{x_1}(\sigma\dot{\epsilon}_\delta(u_\delta+\phi_\delta)^4+k_3(u_\delta+\phi_\delta)-\dot{h}_\delta)|\eta_\delta^2\,dv\\ &\ge -k_2\Big(\int_{\Omega_\delta}(\sigma\dot{\epsilon}_\delta(u_\delta+\phi_\delta)^4+k_3(u_\delta+\phi_\delta)-\dot{h}_\delta)^2\eta_\delta^2\,dv\Big)^{1/2}\\ &\times\Big(\int_{\Omega_\delta}|\nabla(u_\delta+\phi_\delta)|^2\eta_\delta^2\,dv)\Big)^{1/2} \ge -c_{10}\delta^{1/2}. \end{aligned}\tag{4.3}$$

Now, let us estimate the integral

$$-\int_{\Sigma_\delta^s}\sigma\epsilon_\delta|u_\delta+\phi_\delta|^3(u_\delta+\phi_\delta)\psi_\delta\,ds.$$

By using the Gauss formula

$$
\begin{aligned}
-\int_{\Sigma_\delta^s} \sigma\epsilon_\delta |u_\delta+\phi_\delta|^3 (u_\delta+\phi_\delta)\psi_\delta\, ds
&= -\sigma \int_{\Omega_\delta} \Big((u_\delta+\phi_\delta)^4 \psi_\delta \big(\frac{\dot\epsilon_\delta}{\delta} x_3 + \frac{\epsilon^+ - \epsilon^-}{2}\big)\Big)_{x_3} dv \\
&\quad - \int\limits_{\Sigma_\delta' \cup \Sigma_\delta''} \sigma\epsilon_\delta |u_\delta+\phi_\delta|^3 (u_\delta+\phi_\delta)\psi_\delta\, ds
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
&\int_{\Omega_\delta} \big((u_\delta+\phi_\delta)^4 \psi_\delta (\frac{\dot\epsilon_\delta}{\delta} x_3 + \frac{\epsilon^+ - \epsilon^-}{2})\big)_{x_3} dv \\
&\quad = \int_{\Omega_\delta} 4(u_\delta+\phi_\delta)^3 (u_\delta+\phi_\delta)_{x_3} \psi_\delta (\frac{\dot\epsilon_\delta}{\delta} x_3 + \frac{\epsilon^+ - \epsilon^-}{2})\, dv \\
&\quad + \int_{\Omega_\delta} (u_\delta+\phi_\delta)^4 (\psi_\delta)_{x_3} (\frac{\dot\epsilon_\delta}{\delta} x_3 + \frac{\epsilon^+ - \epsilon^-}{2})\, dv + \frac{1}{\delta}\int_{\Omega_\delta} \dot\epsilon_\delta (u_\delta+\phi_\delta)^4 \psi_\delta\, dv.
\end{aligned} \tag{4.5}
$$

If we now take into account uniform boundedness of the functions $u_\delta + \phi_\delta$, h_δ with respect to δ and (3.5), then we get:

$$
\begin{aligned}
&\int_{\Omega_\delta} 4(u_\delta+\phi_\delta)^3 (u_\delta+\phi_\delta)_{x_3} \psi_\delta (\frac{\dot\epsilon_\delta}{\delta} x_3 + \frac{\epsilon^+ - \epsilon^-}{2})\, dv \\
&\geq -\Big(\int_{\Omega_\delta} 16 (\frac{\dot\epsilon_\delta}{\delta} x_3)^2 (u_\delta+\phi_\delta)^6 (\sigma\dot\epsilon_\delta (u_\delta+\phi_\delta)^4 + k_3(u_\delta+\phi_\delta) - \dot h_\delta)^2 \eta_\delta^2\, dv \Big)^{1/2} \\
&\qquad \times \Big(\int_{\Omega_\delta} |\nabla (u_\delta+\phi_\delta)|^2 \eta_\delta^2\, dv \Big)^{1/2} \\
&- \Big(\int_{\Omega_\delta} 16 (\frac{\epsilon^+ - \epsilon^-}{2})^2 (u_\delta+\phi_\delta)^6 (\sigma\dot\epsilon_\delta (u_\delta+\phi_\delta)^4 + k_3(u_\delta+\phi_\delta) - \dot h_\delta)^2 \eta_\delta^2\, dv \Big)^{1/2} \\
&\qquad \times \Big(\int_{\Omega_\delta} |\nabla (u_\delta+\phi_\delta)|^2 \eta_\delta^2\, dv \Big)^{1/2} \geq -c_{11}\delta^{1/2},
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
&\int_{\Omega_\delta} (u_\delta+\phi_\delta)^4 (\psi_\delta)_{x_3} \big(\frac{\dot\epsilon_\delta}{\delta} x_3 + \frac{\epsilon^+ - \epsilon^-}{2}\big)\, dv \\
&\quad \geq -\Big(\int_{\Omega_\delta} (\frac{\dot\epsilon_\delta}{\delta} x_3)^2 (u_\delta+\phi_\delta)^8\, dv \Big)^{1/2} \Big(\int_{\Omega_\delta} |\nabla\psi_\delta|^2\, dv \Big)^{1/2} \\
&- \Big(\int_{\Omega_\delta} (\frac{\epsilon^+ - \epsilon^-}{2})^2 (u_\delta+\phi_\delta)^8\, dv \Big)^{1/2} \Big(\int_{\Omega_\delta} |\nabla\psi_\delta|^2\, dv \Big)^{1/2} \geq -c_{12}\delta^{1/2}
\end{aligned} \tag{4.7}
$$

and

$$
\int\limits_{\Sigma_\delta' \cup \Sigma_\delta''} \sigma\epsilon_\delta |u_\delta+\phi_\delta|^3 (u_\delta+\phi_\delta)\psi_\delta \geq -c_{13}\delta. \tag{4.8}
$$

If we put (4.4) – (4.8) together, then

$$-\int_{\Sigma_\delta^s} \sigma\epsilon_\delta|u_\delta+\phi_\delta|^3(u_\delta+\phi_\delta)\psi_\delta\,ds \le -\frac{1}{\delta}\int_{\Omega_\delta} \sigma\dot\epsilon_\delta(u_\delta+\phi_\delta)^4\psi_\delta\,dv+c_{14}\delta^{1/2}. \quad (4.9)$$

Using similar techniques, we can also get the following estimates:

$$\int_{\Sigma_\delta^s} h_\delta\psi_\delta\,ds \le \frac{1}{\delta}\int_{\Omega_\delta} \dot h_\delta\psi_\delta\,dv + c_{15}\delta^{1/2}, \quad (4.10)$$

$$-\int_{\Sigma_\delta^s} k_3(u_\delta+\phi_\delta)\psi_\delta\,ds \le -\frac{1}{\delta}\int_{\Omega_\delta} k_3(u_\delta+\phi_\delta)\psi_\delta\,dv + c_{16}\delta^{1/2}. \quad (4.11)$$

Now, if we finally take into account (2.10), (4.2), (4.3), (4.9) – (4.11), then we get

$$-c_{17}\delta^{(1-r)/2} \le \frac{1}{\delta}\int_{\Omega_\delta} \big(\dot h_\delta - \sigma\dot\epsilon_\delta(u_\delta+\phi_\delta)^4 - k_3(u_\delta+\phi_\delta)\big)\psi_\delta\,dv$$

or

$$\int_{\Omega_\delta} (\sigma\dot\epsilon_\delta(u_\delta+\phi_\delta)^4 + k_3(u_\delta+\phi_\delta) - \dot h_\delta)^2\eta_\delta^2\,dv \le c_{17}\delta^{(3-r)/2}. \quad (4.12)$$

If we choose a set $\Omega_\delta' := \{x\in\Omega_\delta : \eta_\delta(x)=1\}$, then (4.12) implies

$$\int_{\Omega_\delta'} (\sigma\dot\epsilon_\delta(u_\delta+\phi_\delta)^4 + k_3(u_\delta+\phi_\delta) - \dot h_\delta)^2\,dv \le c_{17}\delta^{(3-r)/2}. \quad (4.13)$$

Since we have $\Omega_\delta\backslash\Omega_\delta' = (0, \frac{l_1\delta^r}{l_3^r}) \times (-l_2, l_2) \times (-\delta,\delta)$, and functions $u_\delta+\phi_\delta$, $\dot h_\delta$ are uniformly bounded with respect to δ, then

$$\int_{\Omega_\delta\backslash\Omega_\delta'} (\sigma\dot\epsilon_\delta(u_\delta+\phi_\delta)^4 + k_3(u_\delta+\phi_\delta) - \dot h_\delta)^2\,dv \le c_{18}\delta^{1+r}. \quad (4.14)$$

If we choose $r=\dfrac{1}{3}$, then estimates (4.13), (4.14) imply

$$\int_{\Omega_\delta} (\sigma\dot\epsilon_\delta(u_\delta+\phi_\delta)^4 + k_3(u_\delta+\phi_\delta) - \dot h_\delta)^2\,dv \le c_{19}\delta^{4/3}. \quad (4.15)$$

5. Final Result

Let us choose functions $\dot\epsilon : Q\mapsto\mathbb{R}$, $a : Q\times\mathbb{R}\mapsto\mathbb{R}$, $\dot f_\delta : Q\mapsto\mathbb{R}$ defined as:

$$\dot\epsilon(x_1,x_2) := \frac{\epsilon^+(x_1,x_2)+\epsilon^-(x_1,x_2)}{2}, \quad a(x_1,x_2,\tau) := \sigma\dot\epsilon(x_1,x_2)|\tau|^3\tau + k_3\tau,$$

$$\dot f_\delta(x_1,x_2) := \frac{f_\delta^+(x_1,x_2)+f_\delta^-(x_1,x_2)}{2}$$

and an operator $\dot{F}_\delta \in \mathfrak{L}(L_2(Q), L_2(Q))$ defined as

$$\dot{F}_\delta := \frac{P_\delta^2(\Sigma_\delta^+)F_\delta P_\delta^1(\Sigma_\delta^+) + P_\delta^2(\Sigma_\delta^-)F_\delta P_\delta^1(\Sigma_\delta^-)}{2}.$$

As $a_\tau(x_1, x_2, \tau) > 0$ for every $(x_1, x_2, \tau) \in Q \times \mathbb{R}$, then the theorem of implicit function guarantees, that there exists a function $b : Q \times \mathbb{R} \mapsto \mathbb{R}$, such that

$$b(x_1, x_2, a(x_1, x_2, \tau)) = \tau,$$

when $(x_1, x_2, \tau) \in Q \times \mathbb{R}$. But then we can define a new function $c : Q \times \mathbb{R} \mapsto \mathbb{R}$ by a mapping

$$c(x_1, x_2, \tau) := \sigma\dot{\epsilon}(x_1, x_2)|b(x_1, x_2, \tau)|^3 b(x_1, x_2, \tau).$$

As $b(x_1, x_2, 0) = 0,\ c(x_1, x_2, 0) = 0$ for every $(x_1, x_2) \in Q$ and

$$|b_\tau(x_1, x_2, \tau)| \leq \frac{1}{k_3}, \quad |c_\tau(x_1, x_2, \tau)| \leq 1$$

for every $(x_1, x_2, \tau) \in Q \times \mathbb{R}$, then, for measurable functions $u : Q \mapsto \mathbb{R}$, $v : Q \mapsto \mathbb{R}$ and almost every $(x_1, x_2) \in Q$ the following estimates will hold:

$$\begin{aligned}
&|b(x_1, x_2, u(x_1, x_2)) - b(x_1, x_2, v(x_1, x_2))| \leq \frac{|u(x_1, x_2) - v(x_1, x_2)|}{k_3},\\
&|c(x_1, x_2, u(x_1, x_2)) - c(x_1, x_2, v(x_1, x_2))| \leq |u(x_1, x_2) - v(x_1, x_2)|,\\
&|b(x_1, x_2, u(x_1, x_2))| \leq \frac{|u(x_1, x_2)|}{k_3},\\
&|c(x_1, x_2, u(x_1, x_2))| \leq |u(x_1, x_2)|.
\end{aligned}$$

But then we get that the operators

$$B(u) := b(x_1, x_2, u(x_1, x_2)), \quad C(u) := c(x_1, x_2, u(x_1, x_2))$$

map $L_2(Q)$ in $L_2(Q)$. Moreover, these operators are Lipschitz continuous.

Now let us consider the following equation

$$v - \dot{F}_\delta(C(v)) = f, \tag{5.1}$$

where $v \in L_2(Q)$ is unknown variable and $f \in L_2(Q)$. Existence and unity of solutions for freely chosen right–hand side $f \in L_2(Q)$ of equation (5.1) easily can be proved by applying the theorem of contractive operators. Indeed, if we choose some functions $v_1 \in L_2(Q)$, $v_2 \in L_2(Q)$, then

$$\begin{aligned}
\|\dot{F}_\delta(C(v_1)) - \dot{F}_\delta(C(v_2))\|_{L_2(Q)} &\leq \|\dot{F}_\delta\|_{\mathfrak{L}(L_2(Q), L_2(Q))}\|C(v_1) - C(v_2)\|_{L_2(Q)}\\
&\leq c_3\|v_1 - v_2\|_{L_2(Q)},
\end{aligned}$$

and $0 \leq c_3 < 1$.

As we know from the theorem of contractive operator, the sequence $\{v^i \in L_2(Q)\}_{i\in\mathbb{N}}$ will converge to the solution v^* of the equation (5.1) (for the fixed $f \in L_2(Q)$), if it is constructed in the following way:

$$\begin{aligned} v^0 &:= f, \\ v^i &:= \dot{F}_\delta(C(v^{i-1})) + f \quad 1 \le i \le \infty. \end{aligned}$$

Now, if we choose some $f_1 \in L_2(Q)$, $f_2 \in L_2(Q)$, then we have:

$$\begin{aligned} \|v_1^1 - v_2^1\|_{L_2(Q)} &= \|\dot{F}_\delta(C(f_1)) - \dot{F}_\delta(C(f_2)) + f_1 - f_2\|_{L_2(Q)} \\ &\le \|\dot{F}_\delta\|_{\mathfrak{L}(L_2(Q),L_2(Q))}\|C(f_1) - C(f_2)\|_{L_2(Q)} + \|f_1 - f_2\|_{L_2(Q)} \\ &\le (c_3 + 1)\|f_1 - f_2\|_{L_2(Q)}, \end{aligned}$$

$$\begin{aligned} \|v_1^2 - v_2^2\|_{L_2(Q)} &= \|\dot{F}_\delta(C(v_1^1)) - \dot{F}_\delta(C(v_2^1)) + f_1 - f_2\|_{L_2(Q)} \\ &\le \|\dot{F}_\delta\|_{\mathfrak{L}(L_2(Q),L_2(Q))}\|C(v_1^1) - C(v_2^1)\|_{L_2(Q)} + \|f_1 - f_2\|_{L_2(Q)} \\ &\le c_3\|v_1^1 - v_2^1\|_{L_2(Q)} + \|f_1 - f_2\|_{L_2(Q)} \le (c_3^2 + c_3 + 1)\|f_1 - f_2\|_{L_2(Q)}, \end{aligned}$$

and therefore

$$\|v_1^* - v_2^*\|_{L_2(Q)} \le \left(\sum_{i=0}^{\infty}(c_3^i)\right)\|f_1 - f_2\|_{L_2(Q)} = c_{20}\|f_1 - f_2\|_{L_2(Q)}.$$

As we found above, there exists an operator $\left(I - \dot{F}_\delta(C)\right)^{-1} : L_2(Q) \mapsto L_2(Q)$ and it is Lipschitz continuous. Therefore, we can define an operator $B((I - \dot{F}_\delta(C))^{-1}) : L_2(Q) \mapsto L_2(Q)$, which is also Lipschitz continuous. Furthermore, it's Lipschitz constant is uniformly bounded with respect to δ.

The operator $B((I - \dot{F}_\delta(C))^{-1})$ can be used to transform (4.15) into the following estimate

$$\int_{\Omega_\delta}(T_\delta - \tilde{T}_\delta)^2\,dv = \int_{\Omega_\delta}\left((u_\delta + \phi_\delta) - \tilde{T}_\delta\right)^2 dv \le c_{21}\delta^{4/3},$$

where the function $\tilde{T}_\delta : \Omega_\delta \mapsto \mathbb{R}$ is defined as

$$\tilde{T}_\delta(x_1, x_2, x_3) := B((I - \dot{F}_\delta(C))^{-1})(\dot{f}_\delta)(x_1, x_2).$$

As we see, $\tilde{T}_\delta$ can be used as a good approximation of temperature T_δ in $L_2(\Omega_\delta)$. We even know the rate of $\tilde{T}_\delta$ convergence to T_δ in $L_2(\Omega_\delta)$ norm.

Finally, it is important to note, that $\tilde{T}_\delta$ is a good temperature approximation even on the surface $\Sigma_\delta^+ \cup \Sigma_\delta^-$ (the proof is similar to previously used):

$$\int_{\Sigma_\delta^+ \cup \Sigma_\delta^-}(T_\delta - \tilde{T}_\delta)^2\,ds \le c_{22}\delta^{1/3}.$$

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Temperatūros modeliavimas tarp plonų medžiagos lakštų atsižvelgiant į radiacijai laidžios šilumos pernešimą

K. Birgelis

Straipsnyje modeliuojamas temperatūros pasiskirstymas tarp plonų medžiagos lakštų atsižvelgiant į radiacijai laidžios šlumos pernešimą. Nustatyta, kad temperatūra tarp lakštų gali būti aproksimuojama L_2 normoje paprastos netiesinės operatorinės lygties sprendiniais.