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ON THE BOUNDARY OF CONVEXITY OF UNIVALENT FUNCTIONS CLASS IN HALF-PLANE

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ABSTRACT

In this article we establish the maximum radius of the disc which any univalent in the half-plane function maps onto a convex domain.

1. INTRODUCTION

Let $K(D)$ be a certain subclass of class of analytical in the domain D functions. Maximal number $R(z_0)$ for which any function of $K(D)$ maps disc with center in the point $z_0 \in D$ and radius $R(z_0)$ onto a convex domain is called the boundary of convexity of class $K(D)$ in the point z_0. The radius problem was raised for various subclasses of analytic in the unit disc functions (see, for instance [3] – [6]). It is known [1], that for class S of functions, which are univalent and normalized $(f(0) = 0,\ f'(0) = 1)$ in the unit disc, the boundary of convexity with respect to point z_0 is the number $r(z_0) = 2 - \sqrt{3 + z_0^2}$. Our goal is to establish the boundary of convexity of class of functions which are univalent and normalized in the half-plane.

2. RESULTS

By S we denote a class of functions $g(\omega) = \omega + b_2\omega^2 + \cdots$ being univalent and normalized in the unit disc $E = \{|\omega| < 1\}$. Let U denote a class of functions of the type

$$F(z) = z - 1 + a_2 (z-1)^2 + \cdots$$

univalent in half-plane $\Pi = \{\operatorname{Re} z > 0\}$. By $C(a, \rho)$ we denote an open disc with a center in point $a \in C$ and with radius ρ.

Main result of the present paper is contained in the following theorem.

Theorem 2.1. (On the boundary of convexity of class U.) *Let z_0 be a fixed point from half-plane Π. If $0 < r \leq \frac{1}{2}\operatorname{Re} z_0$, then any function $F(z) \in U$ maps disc $C(z_0, r)$ onto convex domain. If $\frac{1}{2}\operatorname{Re} z_0 < r < \operatorname{Re} z_0$, then in class U there exists a function mapping disc $C(z_0, r)$ onto non-convex domain.*

To prove the theorem we need to have an auxiliary lemma.

Lemma 2.1. *Let $r < z_0$. Then function $\omega = (z-1)/(z+1)$ maps $C(z_0, r)$ onto $C(\omega_0, R)$, (and function $z = (1+\omega)/(1-\omega)$ consequently, $C(\omega_0, R)$ onto $C(z_0, r)$), where*

$$\omega_0 = \frac{z_0^2 - r^2 - 1}{(z_0+1)^2 - r^2}, \qquad R = \frac{2r}{(z_0+1)^2 - r^2}. \tag{2.1}$$

Proof of this Lemma follows from circular property of linear-fractional mapping and principle of symmetry for analytic functions.

Proof of the theorem 2.1. First, let us consider a case when point $z_0 \in \Pi$ lies on a real positive semi-axis. Let z_0 and r be connected by a relation $z_0 = kr$, $k > 1$. Note that in this case conditions $0 < r \leq z_0/2$ and $z_0/2 < r < z_0$ are equivalent to conditions $k \geq 2$ and $1 < k < 2$ respectively. Let us assume that on condition that $0 < r \leq z_0/2$ (i.e. when $k \geq 2$) in class U function $F_0(z)$ has been found which maps disc $C(z_0, r) \subset \Pi$ on some non-convex domain D. Then function

$$g_0(\omega) = \frac{1}{2} F_0\left(\frac{1+\omega}{1-\omega}\right)$$

belonging to class S would map disc $C(\omega_0, R)$, where ω_0 and R are found from (2.1), onto non-convex domain $\frac{1}{2}D$. In fact, it is obvious that $r < z_0$ and by lemma function

$$z = \frac{1+\omega}{1-\omega}$$

maps $C(\omega_0, R)$ onto $C(z_0, r)$.

For any $r > 0$ and $k > 0$ the following identity takes place

$$R^2 - 4R + 1 - {\omega_0}^2 = \frac{4r(k-2)}{(kr+1)^2 - r^2}, \tag{2.2}$$

where R and ω_0 are defined from relations (2.1) when $z_0 = kr$. If $k \geq 2$ then for any $r > 0$ we have $R^2 - 4R + 1 - \omega_0{}^2 \geq 0$. In this case any function of class S (including also $g_0(\omega)$) should map $C(\omega_0, R)$ onto convex domain [1]. Obtained contradiction shows that on condition that $0 < r \leq z_0/2$ the assumption on existence in class U a function mapping disc $C(z_0, r)$ onto non-convex domain was incorrect.

If $1 < k < 2$ (i.e. $z_0/2 < r < z_0$), then from (2.2) we receive

$$R = 2 - \sqrt{3 + \omega_0{}^2} + \varepsilon, \quad \varepsilon > 0.$$

Since it is obvious that $C(\omega_0, R) \subset E$, then $R < 1 - |\omega_0|$ and, hence

$$0 < \varepsilon < 1 - |\omega_0| - \left(2 - \sqrt{3 + \omega_0{}^2}\right).$$

Thus [1] in class S one can find such a function $g_1(\omega)$, which maps $C(\omega_0, R)$ onto non-convex domain G. But then, function

$$F_1(z) = 2g_1\left(\frac{z-1}{z+1}\right),$$

belonging to class U will map disc $C(z_0, r)$ onto non-convex domain $2G$, since according to lemma, function

$$\omega = \frac{z-1}{z+1}$$

maps $C(z_0, r)$ onto $C(\omega_0, R)$. So, in case when z_0 lies on the positive real semi-axis, this theorem is proved.

Now let $z_0 = x_0 + \mathrm{i}y_0$ be an arbitrary point of Π and $0 < r \leq \frac{1}{2}\mathrm{Re}\, z_0$. As it was proved, any function of U maps disc $C(x_0, r)$ onto convex domain. If there was function $F_2(z)$ in U mapping $C(z_0, r)$ onto non-convex domain, then function

$$F_3(z) = \frac{F_2(z + \mathrm{i}y_0) - F_2(1 + \mathrm{i}y_0)}{F_2'(1 + \mathrm{i}y_0)},$$

also belonging to class u ($F_2'(1 + \mathrm{i}y_0) \neq 0$, since $F_2(z) \in U$ [2]), would map disc $C(x_0, r)$ onto non-convex domain, that is impossible. The first part of theorem, in any case, is proved. Similarly, if $z_0 = x_0 + \mathrm{i}y_0 \in \Pi$ and $\frac{1}{2}\mathrm{Re}\, z_0 < r < \mathrm{Re}\, z_0$, on the basis of the proved, in class U there exists function $F_4(z)$, mapping $C(x_0, r)$ onto non-convex domain. Then function

$$F_5(z) = \frac{F_4(z - \mathrm{i}y_0) - F_4(1 - \mathrm{i}y_0)}{F_4'(1 - \mathrm{i}y_0)},$$

belonging to U will map $C(z_o, r)$ onto non-convex domain. Theorem is fully proved. ■

3. CONCLUSION

In the present paper we have established, that the boundary of convexity with respect to point z_0 of the class U of functions, which are univalent and normalized $(F(1) = 0,\ F'(1) = 1)$ in the half-plane $\Pi = \{\mathrm{Re}\, z > 0\}$ is the number $R(z_0) = \frac{1}{2}\mathrm{Re}\, z_0$, that looks surprisingly simple.

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Apie vienalapių pusplokštumėje funkcijų klasės iškilumo spindulį

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Straipsnyje nustatomas maksimalus spindulys skritulio, kurį visos vienalapės ir normuotos pusplokštumėje funkcijos atvaizduoja į iškiląją sritį. Tegul $K(D)$ – poklasė analizinių srityje D funkcijų klasės. Maksimalų skaičių $R(z_0)$ tokį, kad visos funkcijos iš $K(D)$ atvaizduoja skritulį su centru taške z_0 ir spinduliu $R(z_0)$ į iškiliąją sritį, vadinsime klasės $K(D)$ iškilumo spinduliu taško z_0 atžvilgiu. Iškilumo spindulio problema buvo keliama įvairioms analizinių funkcijų vienetiniame skritulyje poklasėms. Šiame straipsnyje nustatomas vienalapių ir normuotų $(F(1) = 0,\ F'(1) = 1)$ pusplokštumėje $\Pi = \{\mathrm{Re}\, z > 0\}$ funkcijų klasės iškilumo spindulys taško z_0 atžvilgiu.