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Global Solution for the Generalized Anisotropic Navier–Stokes Equations with Large Data

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Abstract. We are concerned with 3D incompressible generalized anisotropic Navier–Stokes equations with hyperdissipative term in horizontal variables. We prove that there exists a unique global solution for it with large initial data in anisotropic Besov space.

Keywords: anisotropic, hyper-dissipation, large data.

AMS Subject Classification: 35Q30; 35Q35; 76D03.

1 Introduction

In this paper, we are going to study the generalized 3-D incompressible anisotropic Navier–Stokes equations

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nu_h D_h^{2\alpha} u + \nu_3 D_3^{2\beta} u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where u and Π represent the velocity and pressure of the fluids respectively. The viscosity coefficients $\nu_h, \nu_3 > 0$. $D_h^{2\alpha}$ and $D_3^{2\beta}$ are two Fourier multipliers whose symbol is $h(\xi_h) = |\xi_h|^{2\alpha}$ and $m(\xi) = |\xi_3|^{2\beta}$ respectively, where $\xi_h \in \mathbb{R}^2$ and $\xi_3 \in \mathbb{R}$. We consider this model in order to understand how much the dissipation we need to dominate the effects of convective term, so that the global solution of such an equation with large data exists. In this way we can understand 3-D Navier–Stokes equations better. In fact, there are some related works on this topic. We remark that when $\nu_h = \nu_3 > 0$ and $\alpha = \beta = 1$, system (1.1) is nothing but the equations of classical Navier–Stokes equations. The global existence of strong solution for it is of course a great open problem, due to the super-critical nature of the equations. An anisotropic case $\alpha=\beta=1$ and $\nu_h>0$, $\nu_3=0$, has been studied first by J.Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier in [3] and D. Iftimie in [6]. In [6], the author proved that such a system is locally wellposed for initial data in the anisotropic Sobolev space $\dot{H}^{0,s}$ for $s>\frac{1}{2}$. Moreover, it has also been proved that if the initial data are small enough in the sense that

$$\|u_0\|_{L^2}^{\varepsilon}\|u_0\|_{\dot{H}^{0,s}}^{1-\varepsilon}\le c\nu_h$$

for some sufficiently small constant c and $\varepsilon=s-\frac{1}{2}$, then the system (1.1) is global wellposed. Furthermore, T. Zhang in [11] proved that when $\alpha=\beta=1$ and $\nu_h>0$, $\nu_3=0$, system (1.1) is global wellposed provide the initial data u_0 satisfies

$$C\nu_h\big\|u_0^h\big\|_{B^{0,\frac{1}{2}}}\exp\big\{C\nu_h^{-4}\big\|u_0^3\big\|_{B^{0,\frac{1}{2}}}^4+1\big\}\le 1,$$

where $B^{0,\frac{1}{2}}$ is the anisotropic Besov space with the regularity in vertical variable. Considering the periodic anisotropic Navier–Stokes equations, M. Paicu obtained global wellposedness in [9].

We note that, all above global results were obtained under the assumptions of small data in some sense. To get result for large data, people tried to strengthen the dissipative symbol $h(\xi_h)$ and $m(\xi_3)$, authors in [7] obtain the global regularity in critical and subcritical hyperdissipation regimes $h(\xi)=|\xi|^\alpha$ for $\alpha\ge\frac{N+2}{4}$, N is the dimension of the space. This corresponds to the case $\alpha=\beta\ge\frac{N+2}{4}$ and $\nu_h=\nu_3>0$ in system (1.1). Tao [10] improved this slightly by establishing global regularity under a slightly weaker condition. He assumes that $h(\xi)\ge|\xi|^{\frac{N+2}{4}}/g(\xi)$ for all sufficiently large ξ and some non-decreasing function $g:\mathbb{R}^+\to\mathbb{R}^+$ such that $\int_0^\infty\frac{1}{sg(s)^4}ds=\infty$. While for the inhomogeneous flows, D. Fang and R. Zi [5] proved the global existence result for the hyperdissipative Navier–Stokes with initial data in subcritical Sobolev spaces.

Just as the classical Navier–Stokes system, (1.1) has a scaling. Indeed, under the following transformation

$$u_\lambda^h(t,x)=\lambda^{2-\frac{1}{\alpha}}u^h\big(\lambda^2t,\lambda^{\frac{1}{\alpha}}x_h,\lambda^{\frac{1}{\beta}}x_3\big),$$
$$u_\lambda^3(t,x)=\lambda^{2-\frac{1}{\beta}}u^3\big(\lambda^2t,\lambda^{\frac{1}{\alpha}}x_h,\lambda^{\frac{1}{\beta}}x_3\big),$$

the scaling of dissipation term is the same as that of convection term. In this sense, the L^2 norm of u_λ^h and u_λ^3 can be given by

$$\big\|u_\lambda^h(t,x)\big\|_{L^2(\mathbb{R}^3)}=\lambda^{2-\frac{2}{\alpha}-\frac{1}{2\beta}}\big\|u^h(t,x)\big\|_{L^2(\mathbb{R}^3)},$$
$$\big\|u_\lambda^3(t,x)\big\|_{L^2(\mathbb{R}^3)}=\lambda^{2-\frac{1}{\alpha}-\frac{3}{2\beta}}\big\|u^3(t,x)\big\|_{L^2(\mathbb{R}^3)}.$$

If one expect that the L^2 energy of u^h be the critical nature of the equation, we need $2-\frac{2}{\alpha}-\frac{1}{2\beta}=0$. Similarly, for u^3, we need $2-\frac{1}{\alpha}-\frac{3}{2\beta}=0$. Obviously, we have the scaling index $\alpha=\beta=\frac{5}{4}$ and it is the case which has been studied in [5, 7, 10]. Here we want to investigate the effects of dissipation of horizontal variables and expect that $\beta=1$. Thus we obtain that $(\alpha,\beta)=(\frac{4}{3},1)$ is critical for the horizontal components u^h, and $(\alpha, \beta) = (2, 1)$ is critical for the vertical component u^3. In this paper, taking advantage of the incompressible condition $\mathrm{div_h} u^h = -\partial_3 u^3$, we can get the global existence result when $(\alpha, \beta) = (\frac{3}{2}, 1)$. One notes that $(\alpha, \beta) = (\frac{3}{2}, 1)$ is the supercritical index for u^3 but the subcritical index for u^h.

More generally, we prove that, for all $\alpha \geq \frac{3}{2}$ and $\beta = 1$, system (1.1) admits a unique global solution for large initial data. It is obviously that for $\alpha \geq \frac{3}{2}$, $\beta = 1$, (1.1) satisfies the following basic energy estimate

$$\big\|u(t)\big\|_{L^2}^2 + 2\nu_h \int_0^t \big\|D_h^\alpha u\big\|_{L^2}^2 d\tau + 2\nu_3 \int_0^t \|\partial_3 u\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2. \tag{1.2}$$

Our main result in this paper concerns the unique solvability of (1.1) with initial data in the anisotropic Besov spaces but without a smallness assumption on u_0.

Theorem 1. *Let $s \geq 0$, $\alpha \geq \frac{3}{2}$, $\beta = 1$, $u_0 \in B^{0,s}(\mathbb{R}^3)$ with* $\operatorname{div} u_0 = 0$. *Then there exists a positive time T, such that* (1.1) *admits a local solution which satisfies*

$$u \in C\big([0,T]; B^{0,s}\big) \quad \textit{and} \quad \nabla_h^\alpha u \in \widetilde{L}^2\big(0,T; B^{0,s}\big), \quad \partial_3 u \in \widetilde{L}^2\big(0,T; B^{0,s}\big).$$

More over, if $s \geq \frac{1}{2}$, then the local solution is unique and can be extended to the global one.

In Theorem 1, we assume that $u_0 \in B^{0,s}(\mathbb{R}^3)$ for all $s \geq 0$. Following the same method, we may find that such result also holds for anisotropic Sobolev space $H^{0,s}$, $s > 0$. We assert that $s > 0$ is necessary when $u_0 \in H^{0,s}$, since Lemma 1 no longer valid for $s = 0$.

Corollary 1. Let $s > 0$, $\alpha \geq \frac{3}{2}$, $\beta = 1$, $u_0 \in H^{0,s}(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. Then there exists a positive time T, such that (1.1) admits a local solution which satisfies

$$u \in C\big([0,T]; H^{0,s}\big) \quad \text{and} \quad \nabla_h^\alpha u \in \widetilde{L}^2\big(0,T; H^{0,s}\big), \quad \partial_3 u \in \widetilde{L}^2(0,T; H^{0,s}).$$

More over, if $s > \frac{1}{2}$, then the local solution is unique and can be extended to the global one.

The above results can be reached through an energy estimate. The advantage of hyper-disspation regime $h(\xi_h) = |\xi_h|^\alpha$ for $\alpha \geq \frac{3}{2}$ in our paper will be revealed in the estimate of convection term. More explicitly, we should split $\|u \cdot \nabla u\|_{L^2}$ into

$$\|u \cdot \nabla u\|_{L^2} \leq \big\|u^h \cdot \nabla_h u\big\|_{L^2} + \big\|u^3 \partial_3 u\big\|_{L^2}.$$

Since $\alpha \geq \frac{3}{2}$, the first term in the right hand side of above inequality can be estimated by

$$\big\|u^h \cdot \nabla_h u\big\|_{L^2} \leq C \big\|\nabla_h^\alpha u\big\|_{L^2}^{\frac{3}{2\alpha}} \|\partial_3 u\|_{L^2}^{\frac{1}{2} - \frac{1}{2\alpha}} \big\|\partial_3 \nabla_h^\alpha u\big\|_{L^2}^{\frac{1}{2\alpha}}.$$

One can see the details in (4.3). Hence one can use Young's inequality to close the energy estimate(see the details in Lemma 2). We observe that when $\alpha = 1$, the convection term was bounded by

$$\|u \cdot \nabla u\|_{L^2} \leq C\|u\|_{L^2}^{\frac{1}{4}}\|\nabla u\|_{L^2}\left\|\nabla^2 u\right\|_{L^2}^{\frac{3}{4}}$$

in three dimensional space, to close the energy estimate, the smallness condition on u_0 is necessary. The second term is more delicate, we will use divergence free on u to write

$$\partial_3 u^3 = -\mathrm{div}_h\, u^h.$$

Another important estimate in our paper is the $B^{0,s}$ energy estimate of convection term when proving the blow-up criterion. It reads that

$$\begin{aligned}\left|\left(\Delta_k^v(u\cdot\nabla u)|\Delta_k^v u\right)_{L^2}\right| \leq{}& Cd_k 2^{-2ks}\|u\|_{L_v^\infty(L_h^2)}^{\frac{1}{2}}\|\nabla_h u\|_{L_v^\infty(L_h^2)}^{\frac{1}{2}}\|u\|_{B^{0,s}}^{\frac{1}{2}}\|\nabla_h u\|_{B^{0,s}}^{\frac{3}{2}}\\ &+ Cd_k 2^{-2ks}\|u\|_{L_v^\infty(L_h^2)}^{\frac{1}{2}}\|\nabla_h u\|_{L_v^\infty(L_h^2)}^{\frac{1}{2}}\|u\|_{B^{0,s}}^{\frac{3}{2}}\|\nabla_h u\|_{B^{0,s}}^{\frac{1}{2}}\\ &+ Cd_k 2^{-2ks}\|\nabla_h u\|_{L_v^\infty(L_h^2)}\|u\|_{B^{0,s}}\|\nabla_h u\|_{B^{0,s}}\end{aligned}$$

for all $s \geq 0$. The similar estimate in Sobolev space H^s was proved in [8] (Lemma 3.1). The main difficulty in this estimate is how to bound the $u^3\partial_3 u$ by the horizonal derivative $\nabla_h u$. To overcome it, we have to use Bony decomposition in vertical direction. We may find that $u^3\partial_3 u$ can be decomposed by

$$\Delta_k^v\left(\sum_{k'\geq k-N_0} S_{k'+2}^v(\partial_3 u)\Delta_{k'}^v u^3\right) + \Delta_k^v\left(\sum_{|k'-k|\leq N_0} S_{k'-2}^v u^3\partial_3\Delta_{k'}^v u\right). \tag{1.3}$$

The first term of (1.3) can be dealt by Bernstein inequality since the spectrum support of $\Delta_{k'}^v u^3$ is contained in an annular. Formally, when such term estimated in Lebesgue space, we can transfer the operator ∂_3 from $S_{k'+2}^v(\partial_3 u)$ to $\Delta_{k'}^v u^3$. Then the divergence free on u enable us to get the desired estimate. But it can not be used in the second term. We will use a commutator estimate and integration by parts to complete the estimate on the second term. One can see the details in Lemma 1.

The rest of the paper is structured as follows. In Section 2, we present the definition of anisotropic Littlewood–Paley decomposition and the theory of anisotropic Besov space. In Section 3, we show the local existence result and prove a blow-up criterion of (1.1). Finally, we present a high order energy estimate of (1.1) and complete the proof the existence of the global solution.

2 Anisotropic Littlewood–Paley Theory

Because the space we will use is anisotropic Besov spaces, so, in this section, we recall the Hölder and Young's inequalities in the frame of anisotropic Lebesgue spaces at first.

Proposition 1. *1) Let $f \in L_v^{r'}(L_h^{p'})$, $g \in L_v^{r''}(L_h^{p''})$ for $1 \le r, r', r'', p, p', p'' \le \infty$, then $fg \in L_v^r(L_h^p)$. And the following inequality holds*

$$\|fg\|_{L_v^r(L_h^p)} \le C\|f\|_{L_v^{r'}(L_h^{p'})}\|g\|_{L_v^{r''}(L_h^{p''})}, \tag{2.1}$$

where $\frac{1}{r} = \frac{1}{r'} + \frac{1}{r''}$ and $\frac{1}{p} = \frac{1}{p'} + \frac{1}{p''}$.

2) Let $f \in L_v^{r'}(L_h^{p'})$, $g \in L_v^{r''}(L_h^{p''})$ for $1 \le r, r', r'', p, p', p'' \le \infty$, then $f \star g \in L_v^r(L_h^p)$. And the following inequality holds

$$\|f \star g\|_{L_v^r(L_h^p)} \le C\|f\|_{L_v^{r'}(L_h^{p'})}\|g\|_{L_v^{r''}(L_h^{p''})}, \tag{2.2}$$

where $1 + \frac{1}{r} = \frac{1}{r'} + \frac{1}{r''}$ and $1 + \frac{1}{p} = \frac{1}{p'} + \frac{1}{p''}$.

To recall the definition of the anisotropic spaces, we have to introduce an anisotropic dyadic decomposition of Fourier variables, which is called the *Anisotropic Littlewood–Paley decomposition.* Let us briefly explain how it may be built.

Let $(\chi(r), \varphi(r))$ be a couple of C^∞ functions satisfying

$$\operatorname{Supp}\chi \subset \left\{|r| \le \frac{4}{3}\right\}, \quad \operatorname{Supp}\varphi \subset \left\{\frac{3}{4} \le |r| \le \frac{8}{3}\right\}, \quad \chi(r) + \sum_{q\in\mathbb{N}}\varphi\bigl(2^{-q}r\bigr) = 1$$

for any $r \in \mathbb{R}$. Let $\varphi_q(r) = \varphi(2^{-q}r)$, $h_q = \mathcal{F}^{-1}\varphi_q$, and $\tilde{h} = \mathcal{F}^{-1}\chi$. The dyadic blocks in vertical frequencies are defined by

$$\Delta_q^v u = 0 \quad \text{if } q < -1, \qquad \Delta_{-1}^v u = \chi(D_3)u = \int_{\mathbb{R}} \tilde{h}(y_3)u(x_h, x_3 - y_3)dy_3,$$

$$\Delta_q^v u = \varphi\bigl(2^{-q}D_3\bigr)u = \int_{\mathbb{R}} h_q(y_3)u(x_h, x_3 - y_3)dy_3 \quad \text{if } q \ge 0.$$

It is also convenient to introduce the following low frequency cut-off:

$$S_q^v u = \sum_{p \le q-1} \Delta_p^v u.$$

Of course, $S_0^v u = \Delta_{-1}^v u$. The anisotropic Littlewood–Paley decomposition has a nice property of quasi-orthogonality:

$$\Delta_p^v \Delta_q^v u \equiv 0 \quad \text{if } |p-q| \ge 2 \quad \text{and} \quad \Delta_p^v\bigl(S_{q-1}^v u \Delta_q^v u\bigr)\bigr) \equiv 0 \quad \text{if } |p-q| \ge 5.$$

Let us give the definition of nonhomogeneous anisotropic Besov spaces.

Definition 1. Let u be a tempered distribution, we set

$$\|u\|_{B_{2,r}^{0,s}} = \left(\sum_{q\ge -1} 2^{rqs}\bigl\|\Delta_q^v u\bigr\|_{L^2}^r\right)^{\frac{1}{r}}$$

with $s \in \mathbb{R}$. We then define the anisotropic Besov space

$$B_{2,r}^{0,s} = \bigl\{u \in \mathcal{S}'\bigl(\mathbb{R}^3\bigr),\ \|u\|_{B_{2,r}^{0,s}} < \infty\bigr\}.$$

Hereafter, we denote that $B^{0,s}_{2,2} := H^{0,s}$, $B^{0,s}_{2,1} := B^{0,s}$. $H^{0,s}$ is the anisotropic Sobolev space. The following two key estimates based on the divergence-free condition.

Proposition 2. *There exists a positive constant C, such that for every divergence-free vector field $u = (u^1, u^2, u^3)$,*

$$\left\|\nabla u^3\right\|_{B^{0,s}} \le C\|\nabla_h u\|_{B^{0,s}}, \quad \left\|\Delta^v_k u\right\|_{L^2_v(L^4_h)} \le d_k 2^{-ks}\|u\|^{\frac12}_{B^{0,s}}\|\nabla_h u\|^{\frac12}_{B^{0,s}}, \tag{2.3}$$

where d_k stands for a generic positive sequence such that $\sum_{k\ge -1} d_k \le 1$.

Proof. The divergence-free condition implies that

$$\nabla u^3 = \big(\nabla_h u^3, -\mathrm{div}_h\, u^h\big),$$

from which the first estimate directly follows. To prove the second estimate, we write

$$\begin{aligned}\left\|\Delta^v_k u\right\|_{L^2_v L^4_h} &\le C\left\|\Delta^v_k\right\|^{\frac12}_{L^2(\mathbb{R}^3)}\left\|\Delta^v_k \nabla_h u\right\|^{\frac12}_{L^2(\mathbb{R}^3)}\\ &\le C\big(2^{-\frac{ks}{2}} d_k^{\frac12}\|u\|^{\frac12}_{B^{0,s}}\big)\big(2^{-\frac{ks}{2}} d_k^{\frac12}\|\nabla_h u\|^{\frac12}_{B^{0,s}}\big)\\ &\le C 2^{-ks} d_k \|u\|^{\frac12}_{B^{0,s}}\|\nabla_h u\|^{\frac12}_{B^{0,s}}. \quad \square\end{aligned} \tag{2.4}$$

The following lemma is the core of the proof of Theorem 1.

Lemma 1. *Let $\alpha \ge \frac32$, then for any real number $s \ge 0$, there exists a positive constant C_s such that, for any u, v, with u divergence free, we have*

$$\begin{aligned}\left|\big(\Delta^v_k(u\cdot\nabla v)|\Delta^v_k v\big)_{L^2}\right| &\le C_s d_k^2 2^{-2ks}\|u\|_{L^\infty_v(L^2_h)}\big(\|v\|^{1-\frac{5}{4\alpha}}_{B^{0,s}}\left\|\nabla^\alpha_h v\right\|^{\frac{5}{4\alpha}}_{B^{0,s}} + \|v\|_{B^{0,s}}\big)\\ &\times\big(\|v\|^{1-\frac1\alpha}_{B^{0,s}}\left\|\nabla^\alpha_h v\right\|^{\frac1\alpha}_{B^{0,s}} + \|v\|_{B^{0,s}}\big) + C_s d_k^2 2^{-2ks}\|v\|_{L^\infty_v(L^2_h)}\\ &\times\big(\|u\|^{1-\frac{5}{4\alpha}}_{B^{0,s}}\left\|\nabla^\alpha_h u\right\|^{\frac{5}{4\alpha}}_{B^{0,s}} + \|u\|_{B^{0,s}}\big)\big(\|v\|^{1-\frac1\alpha}_{B^{0,s}}\left\|\nabla^\alpha_h v\right\|^{\frac1\alpha}_{B^{0,s}} + \|v\|_{B^{0,s}}\big),\end{aligned} \tag{2.5}$$

where $(\cdot|\cdot)_{L^2}$ denotes the L^2 inner product of two functions. In particular, if $v = u$, there holds

$$\begin{aligned}\left|\big(\Delta^v_k(u\cdot\nabla u)|\Delta^v_k u\big)_{L^2}\right| &\le C_s d_k^2 2^{-2ks}\big(\|u\|^{\frac12}_{L^\infty_v(L^2_h)}\|\nabla_h u\|^{\frac12}_{L^\infty_v(L^2_h)}\|u\|^{\frac12}_{B^{0,s}}\|\nabla_h u\|^{\frac32}_{B^{0,s}}\\ &+ \|u\|^{\frac12}_{L^\infty_v(L^2_h)}\|\nabla_h u\|^{\frac12}_{L^\infty_v(L^2_h)}\|u\|^{\frac32}_{B^{0,s}}\|\nabla_h u\|^{\frac12}_{B^{0,s}}\\ &+ \|\nabla_h u\|_{L^\infty_v(L^2_h)}\|u\|_{B^{0,s}}\|\nabla_h u\|_{B^{0,s}}\big).\end{aligned} \tag{2.6}$$

Proof. Let us define

$$F^h_k := \Delta^v_k\big(u^h\cdot\nabla_h v\big) \quad\text{and}\quad F^v_k := \Delta^v_k\big(u^3\partial_3 v\big).$$

The term $(F^h_k|\Delta^v_k v)_{L^2}$ and $(F^v_k|\Delta^v_k v)_{L^2}$ are estimated in different ways. Let us start by proving the result for F^h_k. Applying Bony decomposition in vertical variable, we have

$$\begin{aligned}F^h_k &= \Delta^v_k(T^v_{u^h}\nabla_h v + T^v_{\nabla_h v}u^h + R^v\big(u^h, \nabla_h v\big)\\ &:= F^{h,1}_k + F^{h,2}_k + F^{h,3}_k,\end{aligned}$$

where

$$F_k^{h,1} = \Delta_k^v \sum_{|k-k'|\le N_0} \big(S_{k'-1}^v u^h \Delta_{k'}^v \nabla_h v\big),$$
$$F_k^{h,2} = \Delta_k^v \sum_{|k-k'|\le N_0} \big(S_{k'-1}^v \nabla_h v \Delta_{k'}^v u^h\big), \quad F_k^{h,3} = \Delta_k^v \sum_{k'>k-N_0} \big(\Delta_{k'}^v \nabla_h v \widetilde{\Delta}_{k'}^v u^h\big).$$

According to Proposition 1, together with interpolation inequality, we get

$$\begin{aligned}
&\big(F_k^{h,1}|\Delta_k^v v\big)_{L^2}\\
&\quad\le C \sum_{|k-k'|\le N_0} \big\|S_{k'-1}^v u^h\big\|_{L_v^\infty(L_h^2)} \big\|\Delta_{k'}^v \nabla_h v\big\|_{L_v^2(L_h^{\frac{8}{3}})} \big\|\Delta_k^v v\big\|_{L_v^2(L_h^8)}\\
&\quad\le C\|u\|_{L_v^\infty(L_h^2)} \sum_{|k'-k|\le N_0} \big\|\Delta_{k'}^v v\big\|_{L^2}^{1-\frac{5}{4\alpha}} \big\|\Delta_{k'}^v \nabla_h^\alpha v\big\|_{L^2}^{\frac{5}{4\alpha}} \big\|\Delta_k^v v\big\|_{L_v^2(H^1(\mathbb{R}^2))}\\
&\quad\le C\|u\|_{L_v^\infty(L_h^2)} \sum_{|k'-k|\le N_0} \big\|\Delta_{k'}^v v\big\|_{L^2}^{1-\frac{5}{4\alpha}} \big\|\Delta_{k'}^v \nabla_h^\alpha v\big\|_{L^2}^{\frac{5}{4\alpha}}\\
&\qquad\times \big(\big\|\Delta_k^v v\big\|_{L^2}^{1-\frac{1}{\alpha}} \big\|\Delta_k^v \nabla_h^\alpha v\big\|_{L^2}^{\frac{1}{\alpha}} + \big\|\Delta_k^v v\big\|_{L^2}\big)\\
&\quad\le C d_k^2 2^{-2ks} \|u\|_{L_v^\infty(L_h^2)} \|v\|_{B^{0,s}}^{1-\frac{5}{4\alpha}} \big\|\nabla_h^\alpha v\big\|_{B^{0,s}}^{\frac{5}{4\alpha}} \big(\|v\|_{B^{0,s}}^{1-\frac{1}{\alpha}} \big\|\nabla_h^\alpha v\big\|_{B^{0,s}}^{\frac{1}{\alpha}} + \|v\|_{B^{0,s}}\big).
\end{aligned} \tag{2.7}$$

Similarly computation implies

$$\begin{aligned}
&\big(F_k^{h,2}|\Delta_k^v v\big)_{L^2}\\
&\quad\le C \sum_{|k-k'|\le N_0} \big\|S_{k'-1}^v \nabla_h v\big\|_{L_v^\infty(H^{-1}(\mathbb{R}^2))} \big\|\Delta_{k'}^v u^h\big\|_{L_v^2(H^{\frac{5}{4}}(\mathbb{R}^2))} \big\|\Delta_k^v v\big\|_{L_v^2(H^1(\mathbb{R}^2))}\\
&\quad\le C\|v\|_{L_v^\infty(L_h^2)} \sum_{|k'-k|\le N_0} \big(\big\|\Delta_{k'}^v u\big\|_{L^2}^{1-\frac{5}{4\alpha}} \big\|\Delta_{k'}^v \nabla_h^\alpha u\big\|_{L^2}^{\frac{5}{4\alpha}} + \big\|\Delta_{k'}^v u\big\|_{L^2}\big)\\
&\qquad\times \big(\big\|\Delta_k^v v\big\|_{L^2}^{1-\frac{1}{\alpha}} \big\|\Delta_k^v \nabla_h^\alpha v\big\|_{L^2}^{\frac{1}{\alpha}} + \big\|\Delta_k^v v\big\|_{L^2}\big)\\
&\quad\le C_s d_k^2 2^{-2ks} \|v\|_{L_v^\infty(L_h^2)} \big(\|u\|_{B^{0,s}}^{1-\frac{5}{4\alpha}} \big\|\nabla_h^\alpha u\big\|_{B^{0,s}}^{\frac{5}{4\alpha}} + \|u\|_{B^{0,s}}\big)\\
&\qquad\times \big(\|v\|_{B^{0,s}}^{1-\frac{1}{\alpha}} \big\|\nabla_h^\alpha v\big\|_{B^{0,s}}^{\frac{1}{\alpha}} + \|v\|_{B^{0,s}}\big).
\end{aligned} \tag{2.8}$$

While for $F_k^{h,3}$, by Proposition 1 and interpolation inequality, we have

$$\begin{aligned}
&\big(F_k^{h,3}|\Delta_k^v v\big)_{L^2}\\
&\quad\le C \sum_{k'\ge k-N_0} \big\|\Delta_{k'}^v u^h\big\|_{L_v^\infty(L_h^2)} \big\|\widetilde{\Delta}_{k'}^v \nabla_h v\big\|_{L_v^2(L_h^{\frac{8}{3}})} \big\|\Delta_k^v v\big\|_{L_v^2(L_h^8)}\\
&\quad\le C\|u\|_{L_v^\infty(L_h^2)} \sum_{k'\ge k-N_0} \big\|\widetilde{\Delta}_{k'}^v v\big\|_{L^2}^{1-\frac{5}{4\alpha}} \big\|\widetilde{\Delta}_{k'}^v \nabla_h^\alpha v\big\|_{L^2}^{\frac{5}{4\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&\times \big(\|\Delta_k^v v\|_{L^2}^{1-\frac{1}{\alpha}}\|\Delta_k^v\nabla_h^\alpha v\|_{L^2}^{\frac{1}{\alpha}}+\|\Delta_k^v v\|_{L^2}\big)\\
&\le Cd_k 2^{-2ks}\|u\|_{L_v^\infty(L_h^2)}\sum_{k'\ge k-N_0}2^{(k-k')s}2^{k's}\|\widetilde{\Delta}_{k'}^v v\|_{L^2}^{1-\frac{5}{4\alpha}}\|\widetilde{\Delta}_{k'}^v\nabla_h^\alpha v\|_{L^2}^{\frac{5}{4\alpha}}\\
&\quad\times\big(\|v\|_{B^{0,s}}^{1-\frac{1}{\alpha}}\|\nabla_h^\alpha v\|_{B^{0,s}}^{\frac{1}{\alpha}}+\|v\|_{B^{0,s}}\big)\\
&\le Cd_k^2 2^{-2ks}\|u\|_{L_v^\infty(L_h^2)}\|v\|_{B^{0,s}}^{1-\frac{5}{4\alpha}}\|\nabla_h^\alpha v\|_{B^{0,s}}^{\frac{5}{4\alpha}}\big(\|v\|_{B^{0,s}}^{1-\frac{1}{\alpha}}\|\nabla_h^\alpha v\|_{B^{0,s}}^{\frac{1}{\alpha}}+\|v\|_{B^{0,s}}\big),
\end{aligned}
\tag{2.9}
$$

where we have used $s\ge 0$ in the last inequality. So (2.7), (2.8) and (2.9) imply that (2.5) is proved for the term $(F_k^h|\Delta_k^v v)_{L^2}$. To estimate the term $(F_k^v|\Delta_k^v v)_{L^2}$, let us again use Bony decomposition in the vertical variable:

$$
\Delta_k^v\big(u^3\partial_3 v\big)=F_k^{v,1}+F_k^{v,2}
$$

with

$$
F_k^{v,1}=\Delta_k^v\sum_{k'\ge k-N_0}S_{k'+2}^v(\partial_3 v)\Delta_{k'}^v u^3,\quad F_k^{v,2}=\Delta_k^v\sum_{|k'-k|\le N_0}S_{k'-1}^v u^3\partial_3\Delta_{k'}^v v.
$$

Clearly, we have

$$
\big|\big(F_k^{v,1}|\Delta_k^v v\big)_{L^2}\big|\le C\|\Delta_k^v v\|_{L_v^2(L_h^8)}\|F_k^{v,1}\|_{L_v^2(L_h^{\frac{8}{7}})}. \tag{2.10}
$$

Bernstein's inequality, along with Proposition 1 yields

$$
\begin{aligned}
\|F_k^{v,1}\|_{L_v^2(L_h^{\frac{8}{7}})}&\le C\sum_{k'\ge k-N_0}2^{k'}\|S_{k'+2}^v v\|_{L_v^\infty(L_h^2)}\|\Delta_{k'}^v u^3\|_{L_v^2(L_h^{\frac{8}{3}})}\\
&\le C\|v\|_{L_v^\infty(L_h^2)}\sum_{k'\ge k-N_0}2^{k'}\|\Delta_{k'}^v u^3\|_{L_v^2(L_h^{\frac{8}{3}})}.
\end{aligned}
\tag{2.11}
$$

Here we are going to use in a crucial way the fact that u is a divergence free:

$$
\partial_3 u^3=-\mathrm{div}_h\, u^h.
$$

Thus, we infer that

$$
\begin{aligned}
\|F_k^{v,1}\|_{L_v^2(L_h^{\frac{8}{7}})}&\le C\|v\|_{L_v^\infty(L_h^2)}\sum_{\substack{k'\ge k-N_0\\ k'\ge 0}}\|\Delta_{k'}^v\nabla_h u^h\|_{L_v^2(L_h^{\frac{8}{3}})}\\
&\quad+C\|v\|_{L_v^\infty(L_h^2)}\|\Delta_{-1}^v u^3\|_{L_v^2(L_h^{\frac{8}{3}})}\\
&\le Cd_k 2^{-ks}\|v\|_{L_v^\infty(L_h^2)}\big(\|u\|_{B^{0,s}}^{1-\frac{5}{4\alpha}}\|\nabla_h^\alpha u\|_{B^{0,s}}^{\frac{5}{4\alpha}}+\|u\|_{B^{0,s}}\big).
\end{aligned}
\tag{2.12}
$$

Here we use the fact that when $k'=-1$, the number of k is finite. Then making use of (2.12) in (2.10), one can deduce that

$$
\begin{aligned}
\big|\big(F_k^{v,1}|\Delta_k^v v\big)_{L^2}\big|&\le Cd_k^2 2^{-2ks}\|v\|_{L_v^\infty(L_h^2)}\big(\|u\|_{B^{0,s}}^{1-\frac{5}{4\alpha}}\|\nabla_h^\alpha u\|_{B^{0,s}}^{\frac{5}{4\alpha}}+\|u\|_{B^{0,s}}\big)\\
&\quad\times\big(\|v\|_{B^{0,s}}^{1-\frac{1}{\alpha}}\|\nabla_h^\alpha v\|_{B^{0,s}}^{\frac{1}{\alpha}}+\|v\|_{B^{0,s}}\big).
\end{aligned}
\tag{2.13}
$$

Up to that point, we did not actually use the energy estimate, but only laws of product or Sobolev embedding. The estimate of the term $(F_k^{v,2}|\Delta_k^v v)_{L^2(\mathbb{R}^3)}$ will use in a crucial way the structure of the nonlinearity. First of all, following a computation done in [4], we get that

$$\big(F_k^{v,2}|\Delta_k^v v\big)_{L^2} = \big(S_k^v u^3 \partial_3 \Delta_k^v v|\Delta_k^v v\big)_{L^2} + R_k(u,v)$$

with $R_k(u,v)$ defined by

$$\sum_{|k-k'|\le N_0} \big(\big[\Delta_k^v, S_{k'-2}u^3\big]\partial_3 \Delta_{k'}^v v|\Delta_k^v v\big)_{L^2} + \sum_{|k-k'|\le N_0} (S_k^v - S_{k'-2}^v)u^3 \partial_3 \Delta_{k'}^v v|\Delta_k^v v)_{L^2}.$$

Then, using an integration by parts, we infer that

$$\big(S_k^v u^3 \partial_3 \Delta_k^v v|\Delta_k^v v\big)_{L^2} = -\frac{1}{2}\big(S_k^v\big(\partial_3 u^3\big)\Delta_k^v v|\Delta_k^v v\big)_{L^2}. \tag{2.14}$$

Thanks to the fact that u is divergence free, $(S_k^v u^3 \partial_3 \Delta_k^v v|\Delta_k^v v)_{L^2}$ can be estimated as the term $(F_k^h|\Delta_k^v)_{L^2}$ which appeared above. The third term $(S_k^v - S_{k'-1}^v)u^3\partial_3 \Delta_{k'}^v v|\Delta_k^v v)_{L^2}$ can be estimated exactly as the term $F_k^{v,1}$. So we will mainly give the details of the proof of $([\Delta_k^v, S_{k'-1}^v u^3]\partial_3 \Delta_{k'}^v v|\Delta_k^v v)_{L^2}$. Note that for any function w,

$$\big[\Delta_k^v, S_{k'-1}u^3\big]w(x) = 2^k \int_{\mathbb{R}} h\big(2^k y_3\big)\big(S_{k'-1}^v u^3(x_h,x_3) - S_{k'-1}^v u^3(x_h, x_3-y_3)\big)w(x_h,x_3-y_3)dy_3.$$

Writing a Taylor formula, we get that

$$\big[\Delta_k^v, S_{k'-1}u^3\big]w(x) = \int_{\mathbb{R}\times[0,1]} \widetilde{h}\big(2^k y_3\big)\big(S_{k'-1}^v \partial_3 u^3\big)\big(x_h, x_3 + t(x_3-y_3)\big)w(x_h,x_3-y_3)dy_3dt$$

with $\widetilde{h}(z) = zh(z)$. Thanks to the fact that u is a divergence free, we get

$$\big[\Delta_k^v, S_{k'-1}^v u^3\big]w(x) = -\int_{\mathbb{R}\times[0,1]} \widetilde{h}\big(2^k y_3\big)\big(S_{k'-1}^v \operatorname{div}_h u^h\big)\big(x_h, x_3 + t(x_3-y_3)\big)w(x_h,x_3-y_3)dy_3dt.$$

Now, applying the rules of product of Sobolev spaces on $\mathbb{R}^2$, one can obtain that

$$\big\|\big[\Delta_k^v, S_{k'-1}^v u^3\big]w(x)\big\|_{H^{-1}(\mathbb{R}^2)} \le C\int_{\mathbb{R}} \big|\widetilde{h}\big(2^k y_3\big)\big|\big\|S_{k'-1}^v \operatorname{div}_h u^h\big\|_{L_v^\infty(H^{-1}(\mathbb{R}^2))}\big\|w(\cdot, x_3-y_3)\big\|_{H^{\frac{5}{4}}(\mathbb{R}^2)}dy_3. \tag{2.15}$$

Applying Hölder inequality in (2.15), we deduce that

$$\big\|[\Delta_k^v, S_{k'-1}^v u^3]w(x)\big\|_{L_v^2(H^{-1}(\mathbb{R}^2))} \le C2^{-k}\big\|S_{k'-1}^v u^h\big\|_{L_v^\infty(L_h^2)}\|w\|_{L_v^2(H^{\frac{5}{4}}(\mathbb{R}^2))}. \tag{2.16}$$

Setting $w = \partial_3\Delta_{k'}^v v$ in (2.16), by interpolation inequality, we finally get that

$$\sum_{|k-k'|\le N_0}\big([\Delta_k^v, S_{k'-1}^v u^3]\partial_3\Delta_{k'}^v v|\Delta_k^v v\big)_{L^2} \le Cd_k^2 2^{-2ks}\|u\|_{L_v^\infty(L_h^2)}$$
$$\times\big(\|v\|_{B^{0,s}}^{1-\frac{5}{4\alpha}}\big\|\nabla_h^\alpha v\big\|_{B^{0,s}}^{\frac{5}{4\alpha}} + \|v\|_{B^{0,s}}\big)\big(\|v\|_{B^{0,s}}^{1-\frac{1}{\alpha}}\big\|\nabla_h^\alpha v\big\|_{B^{0,s}}^{\frac{1}{\alpha}} + \|v\|_{B^{0,s}}\big).$$

Thus we conclude the proof of (2.5).

If $v = u$, by Hölder inequality, we infer that

$$\big(F_k^{h,1}|\Delta_k^v u\big)_{L^2} \le \sum_{|k-k'|\le N_0}\big\|\Delta_k\big(S_{k'-1}^v u^h\cdot\nabla_h\Delta_{k'}^v u\big)\big\|_{L_v^2(L_h^{\frac{4}{3}})}\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)}. \tag{2.17}$$

By Proposition 2, we can deduce that

$$\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)} \le d_k 2^{-ks}\|u\|_{B^{0,s}}^{\frac{1}{2}}\|\nabla_h u\|_{B^{0,s}}^{\frac{1}{2}}. \tag{2.18}$$

Using Bernstein inequality and Hölder inequality, we get

$$\begin{aligned}\big\|\Delta_k^v\big(S_{k'-1}u^h\cdot\nabla_h\Delta_{k'}^v u\big)\big\|_{L_v^2(L_h^{\frac{4}{3}})} &\le C\big\|S_{k'-1}^v u^h\big\|_{L_v^\infty(L_h^4)}\big\|\nabla_h\Delta_{k'}^v u\big\|_{L_v^2(L_h^2)}\\ &\le C2^{-k's}d_{k'}\big\|u^h\big\|_{L_v^\infty(L_h^4)}\|\nabla_h u\|_{B^{0,s}}\\ &\le C2^{-k's}d_{k'}\|u\|_{L_v^\infty(L_h^2)}^{\frac{1}{2}}\|\nabla_h u\|_{L_v^\infty(L_h^2)}^{\frac{1}{2}}\|\nabla_h u\|_{B^{0,s}}.\end{aligned} \tag{2.19}$$

Making use of (2.18) and (2.19) in (2.17), one can obtain that

$$\big(F_k^{h,1}|\Delta_k^v u\big)_{L^2} \le Cd_k^2 2^{-2ks}\|u\|_{L_v^\infty(L_h^2)}^{\frac{1}{2}}\|\nabla_h u\|_{L_v^\infty(L_h^2)}^{\frac{1}{2}}\|u\|_{B^{0,s}}^{\frac{1}{2}}\|\nabla_h u\|_{B^{0,s}}^{\frac{3}{2}}.$$

Arguing as in the estimate of term $F_k^{h,1}$, we get

$$\begin{aligned}\big(F_k^{h,2}|\Delta_k^v u\big)_{L^2} &\le \sum_{|k-k'|\le N_0}\big\|\Delta_k^v\big(\Delta_{k'}^v u^h\cdot\nabla_h S_{k'-1}^v u\big)\big\|_{L_v^2(L_h^{\frac{4}{3}})}\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)}\\ &\le C\sum_{|k-k'|\le N_0}\big\|\Delta_{k'}^v u\big\|_{L_v^2(L_h^4)}\|\nabla_h u\|_{L_v^\infty(L_h^2)}\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)}\\ &\le C2^{-2ks}d_k^2\|\nabla_h u\|_{L_v^\infty(L_h^2)}\|u\|_{B^{0,s}}\|\nabla_h u\|_{B^{0,s}}.\end{aligned}$$

Similarly, for the term $F_k^{h,3}$, we have

$$\begin{aligned}\big(F_k^{h,3}|\Delta_k^v u\big)_{L^2} &\le C\sum_{k'\ge k-N_0}\big\|\Delta_{k'}^v u^h\big\|_{L_v^\infty(L_h^4)}\big\|\widetilde{\Delta}_{k'}^v\nabla_h u\big\|_{L_v^2(L_h^2)}\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)}\\ &\le C\sum_{k'\ge k-N_0}\|u\|_{L_v^\infty(L_h^4)}\big\|\widetilde{\Delta}_{k'}^v\nabla_h u\big\|_{L^2}\|u\|_{B^{0,s}}^{\frac{1}{2}}\|\nabla_h u\|_{B^{0,s}}^{\frac{1}{2}}\end{aligned}$$

$$\leq Cd_k^2 2^{-2ks}\|u\|_{L_v^\infty(L_h^4)}\|u\|_{B^{0,s}}^{\frac12}\|\nabla_h u\|_{B^{0,s}}^{\frac12}\sum_{k'\geq k-N_0}2^{(k-k')s}2^{k's}\big\|\widetilde{\Delta}_{k'}^v\nabla_h u\big\|_{L^2}$$

$$\leq Cd_k^2 2^{-2ks}\|u\|_{L_v^\infty(L_h^2)}^{\frac12}\|\nabla_h u\|_{L_v^\infty(L_h^2)}^{\frac12}\|u\|_{B^{0,s}}^{\frac12}\|\nabla_h u\|_{B^{0,s}}^{\frac32}, \tag{2.20}$$

where we have used $s\geq 0$ in the last inequality of (2.20). The vertical term F_k^v is also more delicate. Using Bony decomposition, along the same line as we did in the proof of (2.5), one can write

$$\Delta_k^v\big(u^3\partial_3 u\big)=F_k^{v,1}+F_k^{v,2}+F_k^{v,3}$$

with

$$F_k^{v,1}=\Delta_k^v\sum_{|k'-k|\leq N_0}S_{k'-1}^v(\partial_3 u)\Delta_{k'}^v u^3,\quad F_k^{v,2}=\Delta_k^v\sum_{|k'-k|\leq N_0}S_{k'-1}^v u^3\partial_3\Delta_{k'}^v u,$$

$$F_k^{v,3}=\Delta_k^v\sum_{k'\geq k-N_0}\widetilde{\Delta}_{k'}^v u^3\partial_3\Delta_{k'}^v u.$$

For $F_k^{v,1}$, we easily get that

$$\begin{aligned}\big(F_k^{v,1}|\Delta_k^v u\big)_{L^2}&\leq C\sum_{|k'-k|\leq N_0}\big\|S_{k'-1}^v(\partial_3 u)\Delta_{k'}^v u^3\big\|_{L_v^2(L_h^{\frac43})}\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)}\\&\leq C\sum_{|k'-k|\leq N_0}\big\|\Delta_{k'}^v u^3\big\|_{L_v^2(L_h^2)}\big\|\partial_3 S_{k'-1}^v u\big\|_{L_v^\infty(L_h^4)}\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)}\\&\leq C\sum_{|k'-k|\leq N_0}\big\|\Delta_{k'}^v\partial_3 u^3\big\|_{L_v^2(L_h^2)}\|u\|_{L_v^\infty(L_h^4)}\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)}.\end{aligned}\tag{2.21}$$

Proposition 2 implies that

$$\big(F_k^{v,1}|\Delta_k^v u\big)_{L^2}\leq Cd_k^2 2^{-2ks}\|u\|_{L_v^\infty(L_h^2)}^{\frac12}\|\nabla_h u\|_{L_v^\infty(L_h^2)}^{\frac12}\|u\|_{B^{0,s}}^{\frac12}\|\nabla_h u\|_{B^{0,s}}^{\frac32}.$$

By the same argument as $F_k^{v,1}$, we get

$$\begin{aligned}\big(F_k^{v,3}|\Delta_k^v u\big)_{L^2}&\leq C\sum_{k'>k-N_0}\big\|\widetilde{\Delta}_{k'}^v(\partial_3 u)\Delta_{k'}^v u^3\big\|_{L_v^2(L_h^{\frac43})}\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)}\\&\leq C\sum_{k'>k-N_0}\big\|\Delta_{k'}^v u^3\big\|_{L_v^2(L_h^2)}\big\|\partial_3\widetilde{\Delta}_{k'}^v u\big\|_{L_v^\infty(L_h^4)}\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)}\\&\leq C\sum_{\substack{k'>k-N_0\\k'\geq 0}}\big\|\Delta_{k'}^v\partial_3 u^3\big\|_{L^2}\|u\|_{L_v^\infty(L_h^4)}\big\|\Delta_k^v u\big\|_{L_v^2(L_h^4)}\\&\quad+C\big\|\Delta_{-1}^v u^3\big\|_{L^2}\|u\|_{L_v^\infty(L_h^4)}\|\Delta_k^v u\|_{L_v^2(L_h^4)}.\end{aligned}\tag{2.22}$$

Here, we also use the fact that when $k'=-1$, the number of k is finite. Again, Proposition 2 implies that

$$\begin{aligned}\big(F_k^{v,3}|\Delta_k^v u\big)_{L^2}&\leq Cd_k^2 2^{-2ks}\|u\|_{L_v^\infty(L_h^2)}^{\frac12}\|\nabla_h u\|_{L_v^\infty(L_h^2)}^{\frac12}\|u\|_{B^{0,s}}^{\frac12}\|\nabla_h u\|_{B^{0,s}}^{\frac32}\\&\quad+Cd_k^2 2^{-2ks}\|u\|_{L_v^\infty(L_h^2)}^{\frac12}\|\nabla_h u\|_{L_v^\infty(L_h^2)}^{\frac12}\|u\|_{B^{0,s}}^{\frac32}\|\nabla_h u\|_{B^{0,s}}^{\frac12}.\end{aligned}$$

For the term $F_k^{v,2}$, following the same computation as in [4] (Lemma 2.1), we get that

$$\left(F_k^{v,2}|\Delta_k^v u\right)_{L^2} = \left(S_k^v u^3 \partial_3 \Delta_k^v u|\Delta_k^v u\right)_{L^2} + R_k(u,u) \tag{2.23}$$

with $R_k(u,u)$ defined by

$$\sum_{|k-k'|\le N_0} \left(\left[\Delta_k^v, S_{k'-1}^v u^3\right]\partial_3 \Delta_{k'}^v u|\Delta_k^v u\right)_{L^2} + \sum_{|k-k'|\le N_0} \left(S_k^v - S_{k'-1}^v\right) u^3 \partial_3 \Delta_{k'}^v u|\Delta_k^v u)_{L^2}.$$

Then, using an integration by parts, we infer that

$$\left(S_k^v u^3 \partial_3 \Delta_k^v u|\Delta_k^v u\right)_{L^2} = -\frac{1}{2}\left(S_k^v(\partial_3 u^3)\Delta_k^v u|\Delta_k^v u\right)_{L^2}.$$

Thanks to the fact that u is divergence free, using Proposition 2, we get

$$\begin{aligned}\left(S_k^v u^3 \partial_3 \Delta_k^v u|\Delta_k^v u\right)_{L^2} &\le C\|\nabla_h u\|_{L_v^\infty(L_h^2)}\left\|\Delta_k^v u\right\|^2_{L_v^2(L_h^4)} \\ &\le C2^{-2ks} d_k^2 \|\nabla_h u\|_{L_v^\infty(L_h^2)}\|u\|_{B^{0,s}}\|\nabla_h u\|_{B^{0,s}}.\end{aligned} \tag{2.24}$$

By Hölder inequality, we have

$$\begin{aligned}&\sum_{|k-k'|\le N_0} \left(\left[\Delta_k^v, S_{k'-1}^v u^3\right]\partial_3 \Delta_{k'}^v u|\Delta_k^v u\right)_{L^2} \\ &\quad \le C \sum_{|k-k'|\le N_0} \left\|\left[\Delta_k^v, S_{k'-1}^v u^3\right]\partial_3 \Delta_{k'}^v u\right\|_{L_v^2(L_h^{\frac{4}{3}})}\left\|\Delta_k^v u\right\|_{L_v^2(L_h^4)}.\end{aligned}$$

Using Bernstein inequality and Proposition 2, we infer that

$$\begin{aligned}&\sum_{|k-k'|\le N_0} \left(\left[\Delta_k^v, S_{k'-1}^v u^3\right]\partial_3 \Delta_{k'}^v u|\Delta_k^v u\right)_{L^2} \\ &\le C \sum_{|k-k'|\le N_0} 2^{k'-k}\left\|S_{k'-1}^v \partial_3 u^3\right\|_{L_v^\infty(L_h^2)}\left\|\Delta_{k'}^v u\right\|_{L_v^2(L_h^4)}\left\|\Delta_k^v u\right\|_{L_v^2(L_h^4)} \\ &\le C\|\nabla_h u\|_{L_v^\infty(L_h^2)}\left\|\Delta_{k'}^v u\right\|_{L_v^2(L_h^4)}\left\|\Delta_k^v u\right\|_{L_v^2(L_h^4)} \\ &\le C2^{-2ks} d_k^2 \|\nabla_h u\|_{L_v^\infty(L_h^2)}\|u\|_{B^{0,s}}\|\nabla_h u\|_{B^{0,s}}.\end{aligned} \tag{2.25}$$

Since $(S_k^v - S_{k'-1}^v)u^3$ is localized in frequency in a ring of size 2^q in the vertical variable, we get

$$\begin{aligned}&\left(\left(S_k^v - S_{k'-1}\right)u^3 \partial_3 \Delta_{k'}^v u|\Delta_k^v u\right)_{L^2} \\ &\le C \sum_{|k-k'|\le N_0} 2^{k'-k}\left\|(S_k^v - S_{k'-1}^v)\partial_3 u^3\right\|_{L_v^\infty(L_h^2)}\left\|\Delta_{k'}^v u\right\|_{L_v^2(L_h^4)}\left\|\Delta_k^v u\right\|_{L_v^2(L_h^4)} \\ &\le C\|\nabla_h u\|_{L_v^\infty(L_h^2)}\left\|\Delta_{k'}^v u\right\|_{L_v^2(L_h^4)}\left\|\Delta_k^v u\right\|_{L_v^2(L_h^4)} \\ &\le C2^{-2ks} d_k^2 \|\nabla_h u\|_{L_v^\infty(L_h^2)}\|u\|_{B^{0,s}}\|\nabla_h u\|_{B^{0,s}}.\end{aligned} \tag{2.26}$$

Combing (2.25) with (2.26), we finally obtain that

$$R_k(u,u) \le C2^{-2ks} d_k^2 \|\nabla_h u\|_{L_v^\infty(L_h^2)}\|u\|_{B^{0,s}}\|\nabla_h u\|_{B^{0,s}}. \tag{2.27}$$

Thus (2.21), (2.22), (2.24) and (2.27) imply that the estimate (2.6) holds true.
□

We will also use the so-called Chemin-Lerner type spaces $\widetilde{L}^\rho_T(B^{0,s})$.

Definition 2. For $s \in \mathbb{R}$, and $T > 0$, we set

$$\|u\|_{\widetilde{L}^\rho_T(B^{0,s})} = \sum_{q\geq -1} 2^{qs}\Big(\int_0^T \|\Delta_q^v u(t)\|^\rho_{L^p} dt\Big)^{\frac{1}{\rho}}$$

and denote by $\widetilde{L}^\rho_T(B^{0,s})$ the subset of distributions $u \in \mathcal{S}'([0,T]\times\mathbb{R}^3)$ with finite $\|u\|_{\widetilde{L}^\rho_T(B^{0,s})}$ norm. When $\rho = \infty$, we take $\operatorname{ess\,sup}_{t\in[0,T]} \|\cdot\|$ instead of $\|\cdot\|_{L^\rho[0,T]}$.

3 Local Existence of the Solution

The purpose of this section is to prove the following local wellposedness result:

Theorem 2. *Let $\alpha \geq \frac{3}{2}$, $\beta = 1$, $s \geq 0$, and $u_0 \in B^{0,s}$ be a divergence free vector field. There exist a positive time T and a unique solution u of* (1.1) *defined on* $[0,T]\times\mathbb{R}^3$ *such that*

$$u \in C\big([0,T], B^{0,s}\big) \quad \textit{and} \quad \nabla_h^\alpha u \in \widetilde{L}^2\big(0,T;B^{0,s}\big), \quad \partial_3 u \in \widetilde{L}^2\big(0,T;B^{0,s}\big).$$

Moreover, if the maximal time T^* of existence is finite, then

$$\lim_{t\to T^*}\int_0^t \Big(\|u\|^{\frac{4\alpha-2}{4\alpha-3}}_{L^\infty_v(L^2_h)}\big\|\nabla_h^\alpha u\big\|^{\frac{2}{4\alpha-3}}_{L^\infty_v(L^2_h)} + \|u\|^{\frac{2\alpha-2}{2\alpha-1}}_{L^\infty_v(L^2_h)}\big\|\nabla_h^\alpha u\big\|^{\frac{2}{2\alpha-1}}_{L^\infty_v(L^2_h)}\Big)d\tau = \infty. \quad (3.1)$$

3.1 Local existence result

In order to prove the local existence of solutions to system (1.1) (Theorem 2), we shall use the classical Friderichs' approximate solutions to system (1.1) (for the details, see [1], Chapter 10). To this end, let us define the sequence of operators $(P_n)_{n\in\mathbb{N}}$ by

$$P_n u := \mathcal{F}^{-1}(1_{B_{(0,n)}}\hat{u}),$$

and we define the following approximate system:

$$\begin{cases}\partial_t u_n + P_n(u_n\cdot\nabla u_n) + \nu_h D_h^{2\alpha}u_n - \nu_3\partial_3^2 u_n = -P_n(\nabla \Pi_n),\\ \operatorname{div} u^n = 0,\\ u^n|_{t=0} = u_0^n,\end{cases}$$

where Π_n is given by

$$-(-\Delta)^{-1}\operatorname{div}(u_n\cdot\nabla u_n).$$

Then Cauchy-Lipschitz Theorem gives a solution $u^n_{n\in\mathbb{N}}$ in $C^1([0,T_n^*);L^2)$, T_n^* is the maximal lifespan of u_n. We aim to exhibit a positive lower bound T of

T_n^* and prove the uniform estimates of u_n. For that, we introduce the solution v to the heat equations, that is

$$\begin{cases}\partial_t v + \nu_h D_h^{2\alpha} v - \nu_3 \partial_3^2 v = 0,\\ v|_{t=0} = S_N u_0,\end{cases} \tag{3.2}$$

where $S_N f = \mathcal{F}^{-1}(\chi(2^{-N}|\xi|)\hat{f})$ and N will be determined later. Clearly (3.2) has a unique global solution satisfying

$$v_N \in C\big([0,\infty); B^{0,s}(\mathbb{R}^3)\big), \quad \nabla_h^\alpha v_N \text{ and } \partial_3 v_N \in \widetilde{L}^2\big(0,\infty; B^{0,s}(\mathbb{R}^3)\big).$$

We want to show that if T is small enough, then $\widetilde{u}_n := u_n - v_{n,N}$ is small with the related norm, where $v_{n,N} := P_n v_N$. To this end, we shall discuss the following perturbation system:

$$\begin{cases}\partial_t \widetilde{u}_n + D_h^{2\alpha}\widetilde{u}_n - \partial_3^2 \widetilde{u}_n + \nabla \Pi_n = H,\\ \operatorname{div}\widetilde{u}_n = 0,\\ \widetilde{u}_n|_{t=0} = (\mathrm{Id} - S_N)u_0^n,\end{cases} \tag{3.3}$$

where

$$H = -(\widetilde{u}_n + v_{n,N})\cdot\nabla v_n - (\widetilde{u}_n + v_{n,N})\cdot\nabla\widetilde{u}_n.$$

We shall now be interested in proving that $\widetilde{u}_{n,N}$ exists on a uniform time interval. Applying Δ_k^v to (3.3) and then taking the L^2 inner product of the resulting equation with $\Delta_k^v\widetilde{u}_{n,N}$, we infer that

$$\frac{d}{dt}\big\|\Delta_k^v\widetilde{u}_n\big\|_{L^2}^2 + 2\nu_h\big\|\nabla_h^\alpha\Delta_k^v\widetilde{u}_n\big\|_{L^2}^2 + 2\nu_3\big\|\partial_3\Delta_k^v\widetilde{u}_n\big\|_{L^2}^2 \le C\big|\big\langle\Delta_k^v H, \Delta_k^v\widetilde{u}_n\big\rangle\big|. \tag{3.4}$$

Performing a time integration, multiplying both sides of (3.4) with 2^{ks} and summing on k, we obtain that

$$\begin{aligned}&\|\widetilde{u}_n\|_{B^{0,s}} + \sqrt{\nu_h}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{\widetilde{L}_t^2(B^{0,s})} + \sqrt{\nu_3}\|\partial_3\widetilde{u}_n\|_{\widetilde{L}_t^2(B^{0,s})}\\ &\quad \le \big\|\widetilde{u}_n(0)\big\|_{B^{0,s}} + C\sum_{k\ge -1}\left(2^{2ks}\int_0^t\big|(\Delta_k^v H|\Delta_k^v\widetilde{u}_n)_{L^2}\big|d\tau\right)^{\frac12}.\end{aligned}$$

By Lemma 2.4 and the definition of H, one has

$$\begin{aligned}\big|\big(\Delta_k^v H|\Delta_k^v\widetilde{u}_n\big)_{L^2}\big| &\le \big|\big(\Delta_k^v(\widetilde{u}_n\cdot\nabla\widetilde{u}_n)|\Delta_k^v\widetilde{u}_n\big)_{L^2}\big| + \big|\big(\Delta_k^v(v_{n,N}\cdot\nabla\widetilde{u}_n)|\Delta_k^v\widetilde{u}_n\big)_{L^2}\big|\\ &\quad + \big|\big(\Delta_k^v(\widetilde{u}_n\cdot\nabla v_{n,N})|\Delta_k^v\widetilde{u}_n\big)_{L^2}\big| + \big|\big(\Delta_k^v(v_{n,N}\cdot\nabla v_{n,N})|\Delta_k^v\widetilde{u}_n\big)_{L^2}\big|.\end{aligned} \tag{3.5}$$

For the first two terms in the right hand side of (3.5), we apply Lemma 1 to obtain that

$$\begin{aligned}&\big|\big(\Delta_k^v(\widetilde{u}_n\cdot\nabla\widetilde{u}_n)|\Delta_k^v\widetilde{u}_n\big)_{L^2}\big|\\ &\quad\le d_k^2 2^{-2ks}\|\widetilde{u}_n\|_{L_v^\infty(L_h^2)}\big(\|\widetilde{u}_n\|_{B^{0,s}}^{1-\frac{5}{4\alpha}}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^{\frac{5}{4\alpha}} + \|\widetilde{u}_n\|_{B^{0,s}}\big)\\ &\qquad\times\big(\|\widetilde{u}_n\|_{B^{0,s}}^{1-\frac{1}{\alpha}}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^{\frac{1}{\alpha}} + \|\widetilde{u}_n\|_{B^{0,s}}\big)\\ &\quad\le C_s d_k^2 2^{-2ks}\|\widetilde{u}_n\|_{B^{0,s}}^2\big(\|\widetilde{u}_n\|_{B^{0,s}}^{\frac{8\alpha}{6\alpha-9}} + \|\widetilde{u}_n\|_{B^{0,s}}^{\frac{8\alpha}{6\alpha-5}} + \|\widetilde{u}_n\|_{B^{0,s}}^{\frac{4\alpha}{3\alpha-2}} + \|\widetilde{u}_n\|_{B^{0,s}}^{\frac43}\big)\\ &\qquad + d_k^2 2^{-2ks}\left(\frac{\nu_3}{256}\|\partial_3\widetilde{u}_n\|_{B^{0,s}}^2 + \frac{\nu_h}{256}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^2\right)\end{aligned} \tag{3.6}$$

for $\alpha > \frac{3}{2}$ and

$$\begin{aligned}
&\big|\big(\Delta_k^v(v_{n,N}\cdot\nabla\widetilde{u}_n)|\Delta_k^v\widetilde{u}_n\big)_{L^2}\big| \le C_s d_k^2 2^{-2ks}\|v_{n,N}\|_{L_v^\infty(L_h^2)}\\
&\quad\times\big(\|\widetilde{u}_n\|_{B^{0,s}}^{1-\frac{5}{4\alpha}}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^{\frac{5}{4\alpha}}+\|\widetilde{u}_n\|_{B^{0,s}}\big)\big(\|\widetilde{u}_n\|_{B^{0,s}}^{1-\frac{1}{\alpha}}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^{\frac{1}{\alpha}}+\|\widetilde{u}_n\|_{B^{0,s}}\big)\\
&\quad+C_s d_k^2 2^{-2ks}\|\widetilde{u}_n\|_{L_v^\infty(L_h^2)}\big(\|v_{n,N}\|_{B^{0,s}}^{1-\frac{5}{4\alpha}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{5}{4\alpha}}+\|v_{n,N}\|_{B^{0,s}}\big)\\
&\quad\times\big(\|\widetilde{u}_n\|_{B^{0,s}}^{1-\frac{1}{\alpha}}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^{\frac{1}{\alpha}}+\|\widetilde{u}_n\|_{B^{0,s}}\big)\\
&\le C_s d_k^2 2^{-2ks}\|\widetilde{u}_n\|_{B^{0,s}}^2\big(\|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-9}}\|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-9}}\\
&\quad+\|v_{n,N}\|_{B^{0,s}}^{\frac{1}{2}}\|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{1}{2}}+\|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-5}}\|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-5}}\\
&\quad+\|v_{n,N}\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}}\|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}}+\|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha-5}{3\alpha-2}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{5}{3\alpha-2}}\\
&\quad+\|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha-5}{3\alpha}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{5}{3\alpha}}+\|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{3\alpha-2}}+\|v_{n,N}\|_{B^{0,s}}^{\frac{4}{3}}\big)\\
&\quad+C_s d_k^2 2^{-2ks}\bigg(\frac{\nu_h}{256}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^2+\frac{\nu_3}{256}\|\partial_3\widetilde{u}_n\|_{B^{0,s}}^2\bigg),
\end{aligned}\tag{3.7}$$

where the sequence d_k satisfies that $\|d_k\|_{l^1}\le 1$. For the third term, we write

$$\begin{aligned}
\big(\Delta_k^v(\widetilde{u}_n\cdot\nabla v_{n,N})|\Delta_k^v\widetilde{u}_n\big)_{L^2} &= \big(\Delta_k^v\big(\widetilde{u}_n^h\cdot\nabla_h v_{n,N}\big)|\Delta_k^v\widetilde{u}_n\big)_{L^2}\\
&\quad+\big(\Delta_k^v\big(\widetilde{u}_n^3\partial_3 v_{n,N}\big)|\Delta_k^v\widetilde{u}_n\big)_{L^2}.
\end{aligned}\tag{3.8}$$

By the same method as in (2.7), (2.8), (2.9) and (3.7), we have the following estimate

$$\begin{aligned}
&\big|\big(\Delta_k^v\big(\widetilde{u}_n^h\cdot\nabla_h v_{n,N}\big)|\Delta_k^v\widetilde{u}_n\big)_{L^2}\big|\\
&\le C_s d_k^2 2^{-2ks}\|v_{n,N}\|_{L_v^\infty(L_h^2)}\big(\|\widetilde{u}_n\|_{B^{0,s}}^{1-\frac{5}{4\alpha}}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^{\frac{5}{4\alpha}}+\|\widetilde{u}_n\|_{B^{0,s}}\big)\\
&\quad\times\big(\|\widetilde{u}_n\|_{B^{0,s}}^{1-\frac{1}{\alpha}}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^{\frac{1}{\alpha}}+\|\widetilde{u}_n\|_{B^{0,s}}\big)\\
&\quad+C_s d_k^2 2^{-2ks}\|\widetilde{u}_n\|_{L_v^\infty(L_h^2)}\big(\|v_{n,N}\|_{B^{0,s}}^{1-\frac{5}{4\alpha}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{5}{4\alpha}}+\|v_{n,N}\|_{B^{0,s}}\big)\\
&\quad\times\big(\|\widetilde{u}_n\|_{B^{0,s}}^{1-\frac{1}{\alpha}}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^{\frac{1}{\alpha}}+\|\widetilde{u}_n\|_{B^{0,s}}\big)\\
&\le C_s d_k^2 2^{-2ks}\|\widetilde{u}_n\|_{B^{0,s}}^2\big(\|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-9}}\|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-9}}+\|v_{n,N}\|_{B^{0,s}}^{\frac{1}{2}}\|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{1}{2}}\\
&\quad+\|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-5}}\|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-5}}+\|v_{n,N}\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}}\|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}}\\
&\quad+\|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha-5}{3\alpha-2}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{5}{3\alpha-2}}+\|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha-5}{3\alpha}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{5}{3\alpha}}+\|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{3\alpha-2}}\\
&\quad+\|v_{n,N}\|_{B^{0,s}}^{\frac{4}{3}}\big)+C_s d_k^2 2^{-2ks}\bigg(\frac{\nu_h}{256}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^2+\frac{\nu_3}{256}\|\partial_3\widetilde{u}_n\|_{B^{0,s}}^2\bigg).
\end{aligned}\tag{3.9}$$

The second term of (3.8) can be estimated by the same method as Lemma 1. More precisely, by the Bony decomposition, we have

$$\begin{aligned}
\Delta_k^v(\widetilde{u}_n\partial_3 v_{n,N}) &= \sum_{|k-k'|\le N_0}\Delta_k^v\big(S_{k'-1}^v\partial_3 v_{n,N}\Delta_{k'}^v\widetilde{u}_n^3\big)\\
&\quad+\sum_{|k-k'|\le N_0}\Delta_k^v\big(S_{k'-1}^v\widetilde{u}_n^3\Delta_{k'}^v\partial_3 v_{n,N}\big)+\sum_{k'>k-N_0}\Delta_k^v\big(\Delta_{k'}^v\widetilde{u}_n^3\widetilde{\Delta}_{k'}^v\partial_3 v_{n,N}\big).
\end{aligned}$$

By the Proposition 1 in anisotropic space, one gets

$$\sum_{|k-k'|\le N_0}\big(\Delta_k^v\big(S_{k'-1}^v\widetilde{u}_n^3\Delta_{k'}^v\partial_3 v_{n,N}\big)|\Delta_k^v\widetilde{u}_n\big)_{L^2}$$
$$\le C\sum_{|k-k'|\le N_0}\big\|S_{k'-1}^v\widetilde{u}_n^3\big\|_{L_v^\infty(L_h^2)}\big\|\Delta_{k'}^v\partial_3 v_{n,N}\big\|_{L_v^2(L_h^{\frac{8}{3}})}\big\|\Delta_k^v\widetilde{u}_n\big\|_{L_v^2(L_h^8)}.$$

Since $v_{n,N}$ is localized in the ξ_3 direction, we can apply Bernstein inequality to obtain

$$\big\|\Delta_{k'}^v\partial_3 v_{n,N}\big\|_{L_v^2(L_h^{\frac{8}{3}})}\le C2^N\big\|\Delta_{k'}^v v_{n,N}\big\|_{L_v^2(L_h^{\frac{8}{3}})}.$$

Thus, interpolation and Young's inequalities imply that

$$\begin{aligned}
&\sum_{|k-k'|\le N_0}\big(\Delta_k^v\big(S_{k'-1}^v\widetilde{u}_n^3\Delta_{k'}^v\partial_3 v_{n,N}\big)|\Delta_k^v\widetilde{u}_n\big)_{L^2}\\
&\le C2^N\|\widetilde{u}_n\|_{L_v^\infty(L_h^2)}\sum_{|k'-k|\le N_0}\big\|\Delta_{k'}^v v_{n,N}\big\|_{L^2}^{1-\frac{1}{4\alpha}}\big\|\Delta_{k'}^v\nabla_h^\alpha v_{n,N}\big\|_{L^2}^{\frac{1}{4\alpha}}\\
&\quad\times\big\|\Delta_k^v\widetilde{u}_n\big\|_{L^2}^{1-\frac{3}{4\alpha}}\big\|\Delta_k^v\nabla_h^\alpha\widetilde{u}_n\big\|_{L^2}^{\frac{3}{4\alpha}}\\
&\le Cd_k^2 2^{-2ks}2^N\|\widetilde{u}_n\|_{B^{0,s}}^{\frac12}\|\partial_3\widetilde{u}_n\|_{B^{0,s}}^{\frac12}\|v_{n,N}\|_{B^{0,s}}^{1-\frac{1}{4\alpha}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{1}{4\alpha}}\\
&\quad\times\big(\|\widetilde{u}_n\|_{B^{0,s}}^{1-\frac{3}{4\alpha}}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^{\frac{3}{4\alpha}}\big)\\
&\le Cd_k^2 2^{-2ks}2^{\frac{8N\alpha}{6\alpha-3}}\|\widetilde{u}_n\|_{B^{0,s}}^2\|v_{n,N}\|_{B^{0,s}}^{\frac{8\alpha-2}{6\alpha-3}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{2}{6\alpha-3}}\\
&\quad+C_s d_k^2 2^{-2ks}\left(\frac{\nu_h}{256}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^2+\frac{\nu_3}{256}\|\partial_3\widetilde{u}_n\|_{B^{0,s}}^2\right).
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&\sum_{|k-k'|\le N_0}\big(\Delta_k^v\big(S_{k'-1}^v\partial_3 v_{n,N}\Delta_{k'}^v\widetilde{u}_{n,N}^3\big)|\Delta_k^v\widetilde{u}_n\big)_{L^2}\\
&\le C2^N\|v_{n,N}\|_{L_v^\infty(L_h^2)}\sum_{|k'-k|\le N_0}\big\|\Delta_{k'}^v\widetilde{u}_n^3\big\|_{L^2}^{1-\frac{1}{4\alpha}}\big\|\Delta_{k'}^v\nabla_h^\alpha\widetilde{u}_n^3\big\|_{L^2}^{\frac{1}{4\alpha}}\\
&\quad\times\big\|\Delta_k^v\widetilde{u}_n\big\|_{L^2}^{1-\frac{3}{4\alpha}}\big\|\Delta_k^v\nabla_h^\alpha\widetilde{u}_n\big\|_{L^2}^{\frac{3}{4\alpha}}\\
&\le Cd_k^2 2^{-2ks}2^N\|v_{n,N}\|_{B^{0,s}}^{\frac12}\|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac12}\|\widetilde{u}_n\|_{B^{0,s}}^{2-\frac{1}{\alpha}}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^{\frac{1}{\alpha}}\\
&\le Cd_k^2 2^{-2ks}2^{\frac{2N\alpha}{2\alpha-1}}\|\widetilde{u}_n\|_{B^{0,s}}^2\|v_{n,N}\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}}\\
&\quad+C_s d_k^2 2^{-2ks}\frac{\nu_h}{256}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^2
\end{aligned}$$

and

$$\begin{aligned}
&*\sum_{k'>k-N_0}\big(\Delta_k^v(\Delta_{k'}^v\widetilde{u}_n^3\widetilde{\Delta}_{k'}^v\partial_3 v_{n,N})|\Delta_k^v\widetilde{u}_n\big)_{L^2}\\
&\quad\le Cd_k^2 2^{-2ks}2^{\frac{8N\alpha}{6\alpha-3}}\|\widetilde{u}_n\|_{B^{0,s}}^2\|v_{n,N}\|_{B^{0,s}}^{\frac{8\alpha-2}{6\alpha-3}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{2}{6\alpha-3}}\\
&\qquad+C_s d_k^2 2^{-2ks}\left(\frac{\nu_h}{256}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^2+\frac{\nu_3}{256}\|\partial_3\widetilde{u}_n\|_{B^{0,s}}^2\right).
\end{aligned}$$

What remains to estimate, therefore, is the term $(\Delta_k^v(v_{n,N}\cdot\nabla v_{n,N})|\Delta_k^v\widetilde{u}_n)_{L^2}$. Since the vector field $v_{n,N}$ is divergence free, we can write

$$\big(\Delta_k^v(v_{n,N}\cdot\nabla v_{n,N})|\Delta_k^v\widetilde{u}_n\big)_{L^2} = -\big(\Delta_k^v(v_{n,N}\otimes v_{n,N})|\Delta_k^v\nabla_h\widetilde{u}_n\big)_{L^2} + \big(\partial_3\Delta_k^v(v_{n,N}\otimes v_{n,N})|\Delta_k^v\widetilde{u}_n\big)_{L^2}.$$

Again, by Bony decomposition we have

$$\Delta_k^v(v_{n,N}\otimes v_{n,N}) = 2\sum_{|k-k'|\le N_0}\Delta_k^v\big(S_{k'-1}^v v_{n,N}\otimes\Delta_{k'}^v v_{n,N}\big) + \sum_{k'>k-N_0}\Delta_k^v\big(\Delta_{k'}^v v_{n,N}\otimes\widetilde{\Delta}_{k'}^v v_{n,N}\big).$$

Thus we have the estimate

$$\begin{aligned}
&\sum_{|k-k'|\le N_0}\big(\Delta_k^v\big(S_{k'-1}^v v_{n,N}\otimes\Delta_{k'}^v v_{n,N}\big)|\Delta_k^v\nabla_h\widetilde{u}_n\big)_{L^2}\\
&\le C\sum_{|k-k'|\le N_0}\big\|S_{k'-1}^v v_{n,N}\big\|_{L_v^\infty(L_h^2)}\big\|\Delta_{k'}^v v_{n,N}\big\|_{L_v^2(L_h^{\frac{2}{\alpha-1}})}\big\|\Delta_k^v\nabla_h\widetilde{u}_n\big\|_{L_v^2(L_h^{\frac{2}{2-\alpha}})}\\
&\le C\|v_{n,N}\|_{L_v^\infty(L_h^2)}\sum_{|k'-k|\le N_0}\big\|\Delta_{k'}^v v\big\|_{L^2}^{2-\frac{2}{\alpha}}\big\|\Delta_{k'}^v\nabla_h^\alpha v\big\|_{L^2}^{\frac{2}{\alpha}-1}\big\|\Delta_k^v\nabla_h^\alpha\widetilde{u}_n\big\|_{L^2}\\
&\le Cd_k^2 2^{-2ks}\|v_{n,N}\|_{B^{0,s}}^{\frac12}\|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac12}\|v_{n,N}\|_{B^{0,s}}^{2-\frac{2}{\alpha}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{2}{\alpha}-1}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}\\
&\le Cd_k^2 2^{-2ks}\bigg(\|v_{n,N}\|_{B^{0,s}}^{5-\frac{4}{\alpha}}\|\partial_3 v_{n,N}\|_{B^{0,s}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{4}{\alpha}-2} + \frac{\nu_h}{256}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^2\bigg)
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&\sum_{k'>k-N_0}\big(\Delta_k^v\big(\Delta_{k'-1}^v v_{n,N}\otimes\widetilde{\Delta}_{k'}^v v_{n,N}\big)|\Delta_k^v\nabla_h\widetilde{u}_n\big)_{L^2}\\
&\le Cd_k^2 2^{-2ks}\Big(\|v_{n,N}\|_{B^{0,s}}^{5-\frac{4}{\alpha}}\|\partial_3 v_{n,N}\|_{B^{0,s}}\big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{4}{\alpha}-2} + \frac{\nu_h}{256}\big\|\nabla_h^\alpha\widetilde{u}_n\big\|_{B^{0,s}}^2\Big).
\end{aligned} \tag{3.11}$$

According the same method as in (3.10), (3.11) and the fact that $v_{n,N}$ is localized in the ξ_3 direction, one has

$$\begin{aligned}
&\sum_{|k-k'|\le N_0}\big(\Delta_k^v\partial_3\big(S_{k'-1}^v v_{n,N}\otimes\Delta_{k'}^v v_{n,N}\big)|\Delta_k^v\widetilde{u}_n\big)_{L^2}\\
&\le C2^N\sum_{|k-k'|\le N_0}\big\|S_{k'-1}^v v_{n,N}\big\|_{L_v^\infty(L_h^2)}\big\|\Delta_{k'}^v v_{n,N}\big\|_{L_v^2(L_h^{\frac83})}\big\|\Delta_k^v\widetilde{u}_n\big\|_{L_v^2(L_h^8)}\\
&\le C2^N\|v_{n,N}\|_{L_v^\infty(L_h^2)}\sum_{|k'-k|\le N_0}\big\|\Delta_{k'}^v v\big\|_{L^2}^{1-\frac{1}{4\alpha}}\big\|\Delta_{k'}^v\nabla_h^\alpha v\big\|_{L^2}^{\frac{1}{4\alpha}}\\
&\quad\times\big\|\Delta_k^v\widetilde{u}_{n,N}\big\|_{L^2}^{1-\frac{3}{4\alpha}}\big\|\Delta_k^v\nabla_h^\alpha\widetilde{u}_{n,N}\big\|_{L^2}^{\frac{3}{4\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&\le C d_k^2 2^{-2ks} 2^N \|v_{n,N}\|_{B^{0,s}}^{\frac12} \|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac12} \|v_{n,N}\|_{B^{0,s}}^{1-\frac{1}{4\alpha}} \big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{1}{4\alpha}} \\
&\quad \times \|\widetilde{u}_n\|_{B^{0,s}}^{1-\frac{3}{4\alpha}} \big\|\nabla_h^\alpha \widetilde{u}_n\big\|_{B^{0,s}}^{\frac{3}{4\alpha}} \\
&\le C_N d_k^2 2^{-2ks} \Big(\|v_{n,N}\|_{B^{0,s}}^{\frac{12\alpha-2}{8\alpha-3}} \|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-3}} \big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{2}{8\alpha-3}} \|\widetilde{u}_n\|_{B^{0,s}}^{\frac{8\alpha-6}{8\alpha-3}} \\
&\quad + \frac{\nu_h}{256} \big\|\nabla_h^\alpha \widetilde{u}_n\big\|_{B^{0,s}}^2 \Big) \qquad (3.12)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k'>k-N_0} \big(\Delta_k^v \partial_3 \big(\Delta_{k'-1}^v v_{n,N} \otimes \widetilde{\Delta}_{k'}^v v_{n,N}\big) | \Delta_k^v \widetilde{u}_n\big)_{L^2} \\
&\quad \le C_N d_k^2 2^{-2ks} \Big(\|v_{n,N}\|_{B^{0,s}}^{\frac{12\alpha-2}{8\alpha-3}} \|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-3}} \big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{2}{8\alpha-3}} \|\widetilde{u}_n\|_{B^{0,s}}^{\frac{8\alpha-6}{8\alpha-3}} \\
&\qquad + \frac{\nu_h}{256} \big\|\nabla_h^\alpha \widetilde{u}_n\big\|_{B^{0,s}}^2 \Big). \qquad (3.13)
\end{aligned}
$$

To summarize, we combine above estimates (3.6)–(3.13) to obtain

$$
\begin{aligned}
&\|\widetilde{u}_n\|_{\widetilde{L}_t^\infty(B^{0,s})} + \sqrt{\nu_h} \big\|\nabla_h^\alpha \widetilde{u}_n\big\|_{\widetilde{L}_t^2(B^{0,s})} + \sqrt{\nu_3} \|\partial_3 \widetilde{u}_n\|_{\widetilde{L}_t^2(B^{0,s})} \\
&\quad \le \big\|\widetilde{u}_n(0)\big\|_{B^{0,s}} + C_N \Big(\int_0^t \|\widetilde{u}_n\|_{B^{0,s}}^2 \big(\|\widetilde{u}_n\|_{B^{0,s}}^{\frac{8\alpha}{6\alpha-9}} + \|\widetilde{u}_n\|_{B^{0,s}}^{\frac{8\alpha}{6\alpha-5}} \\
&\qquad + \|\widetilde{u}_n\|_{B^{0,s}}^{\frac{4\alpha}{3\alpha-2}} + \|\widetilde{u}_n\|_{B^{0,s}}^{\frac43} + \|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-9}} \|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-9}} \\
&\qquad + \|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-5}} \|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-5}} + \|v_{n,N}\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}} \|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}} \\
&\qquad + \|v_{n,N}\|_{B^{0,s}}^{\frac12} \|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac12} + \|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha-5}{3\alpha-2}} \big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{5}{3\alpha-2}} \\
&\qquad + \|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha-5}{3\alpha}} \big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{5}{3\alpha}} + \|v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{3\alpha-2}} + \|v_{n,N}\|_{B^{0,s}}^{\frac43} \\
&\qquad + \|v_{n,N}\|_{B^{0,s}}^{\frac{8\alpha-2}{6\alpha-3}} \big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{2}{6\alpha-3}} + \|v_{n,N}\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}} \big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}} \big) \\
&\qquad + \|v_{n,N}\|_{B^{0,s}}^{5-\frac{4}{\alpha}} \|\partial_3 v_{n,N}\|_{B^{0,s}} \big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{4}{\alpha}-2} \\
&\qquad + \|\widetilde{u}_n\|_{B^{0,s}}^{\frac{8\alpha-6}{8\alpha-3}} \|v_{n,N}\|_{B^{0,s}}^{\frac{12\alpha-2}{8\alpha-3}} \|\partial_3 v_{n,N}\|_{B^{0,s}}^{\frac{4\alpha}{8\alpha-3}} \big\|\nabla_h^\alpha v_{n,N}\big\|_{B^{0,s}}^{\frac{2}{8\alpha-3}} d\tau \Big)^{\frac12}.
\end{aligned}
$$

Then we assume that for all $t \in [0, T]$

$$\big\|\widetilde{u}_n(t)\big\|_{B^{0,s}} \le 2c.$$

Here T will be determined later. We want to prove that the above inequality holds strictly on $[0, T]$. Taking N sufficiently large such that

$$\big\|(\mathrm{Id} - S_N) u_0^n\big\|_{B^{0,s}} \le c,$$

then, for the fixed N, using the above estimate, one gets that

$$
\begin{aligned}
&\|\widetilde{u}_n\|_{\widetilde{L}_t^\infty(B^{0,s})} + \sqrt{\nu_h} \big\|\nabla_h^\alpha \widetilde{u}_n\big\|_{\widetilde{L}_t^2(B^{0,s})} + \sqrt{\nu_3} \|\partial_3 \widetilde{u}_n\|_{\widetilde{L}_t^2(B^{0,s})} \\
&\quad \le c + C_N \big(4c^2 T \ + 4c^2 T^{\frac{6\alpha-9}{8\alpha-9}} \|u_0\|_{B^{0,s}}^4 + T^{\frac{3\alpha-4}{2\alpha}} \|u_0\|_{B^{0,s}}^4 \big)^{\frac12}
\end{aligned}
$$

for $\alpha > \frac{3}{2}$. While for $\alpha = \frac{3}{2}$, we just take the value of α in (3.6)–(3.13) except the estimate (3.6). Indeed, (3.6) can be rewritten as

$$\begin{aligned}
&\big|\big(\Delta_k^v(\widetilde{u}_n \cdot \nabla \widetilde{u}_n)|\Delta_k^v \widetilde{u}_n\big)_{L^2}\big| \\
&\quad \le d_k^2 2^{-2ks} \|\widetilde{u}_n\|_{L_v^\infty(L_h^2)} \big(\|\widetilde{u}_n\|_{B^{0,s}}^{\frac{1}{6}} \big\|\nabla_h^\alpha \widetilde{u}_n\big\|_{B^{0,s}}^{\frac{5}{6}} + \|\widetilde{u}_n\|_{B^{0,s}}\big) \\
&\qquad \times \big(\|\widetilde{u}_n\|_{B^{0,s}}^{\frac{1}{3}} \big\|\nabla_h^\alpha \widetilde{u}_n\big\|_{B^{0,s}}^{\frac{2}{3}} + \|\widetilde{u}_n\|_{B^{0,s}}\big) \\
&\quad \le C_s d_k^2 2^{-2ks} \|\widetilde{u}_n\|_{B^{0,s}}^2 \big(\|\widetilde{u}_n\|_{B^{0,s}}^2 + \|\widetilde{u}_n\|_{B^{0,s}}^3 + \|\widetilde{u}_n\|_{B^{0,s}}^{\frac{12}{5}} + \|\widetilde{u}_n\|_{B^{0,s}}^{\frac{4}{3}}\big) \\
&\qquad + d_k^2 2^{-2ks} \left(\frac{\nu_3}{256} \|\partial_3 \widetilde{u}_n\|_{B^{0,s}}^2 + \frac{\nu_h}{256} \big\|\nabla_h^\alpha \widetilde{u}_n\big\|_{B^{0,s}}^2\right).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\|\widetilde{u}_n\|_{\widetilde{L}_t^\infty(B^{0,s})} + \sqrt{\nu_h} \big\|\nabla_h^\alpha \widetilde{u}_n\big\|_{\widetilde{L}_t^2(B^{0,s})} + \sqrt{\nu_3} \|\partial_3 \widetilde{u}_n\|_{\widetilde{L}_t^2(B^{0,s})} \\
&\quad \le c + C_N \big(4c^2 T \ + 4c^2 T^{\frac{4}{9}} \|u_0\|_{B^{0,s}}^4 + T^{\frac{1}{6}} \|u_0\|_{B^{0,s}}^4\big)^{\frac{1}{2}}
\end{aligned}$$

for $\alpha = \frac{3}{2}$. Here, without loss generality, we assume that $T \le 1, c \le \frac{1}{4}$ and $\|u_0\|_{B^{0,s}} \ge 1$. For the fixed c, and N which depends on c, we select a positive real number T such that

$$C_N \big(4c^2 T \ + 4c^2 T^{\frac{6\alpha-9}{8\alpha-9}} \|u_0\|_{B^{0,s}}^4 + T^{\frac{3\alpha-4}{2\alpha}} \|u_0\|_{B^{0,s}}^4\big)^{\frac{1}{2}} \le \frac{1}{2} c.$$

While for $\alpha = \frac{3}{2}$, we assume that

$$C_N \big(4c^2 T \ + 4c^2 T^{\frac{4}{9}} \|u_0\|_{B^{0,s}}^4 + T^{\frac{1}{6}} \|u_0\|_{B^{0,s}}^4\big)^{\frac{1}{2}} \le \frac{1}{2} c.$$

Then we have

$$\big\|\widetilde{u}_n(t)\big\|_{B^{0,s}} \le \frac{3}{2} c, \quad \forall t \in [0, T].$$

This implies that $\widetilde{u}_n$ exists on a uniform time interval $[0, T]$, T is independent of n. A standard compactness argument which based on the Ascoli's theorem implies that u^n tends to u which satisfies (1.1). Moreover

$$u \in \widetilde{L}^\infty\big(0, T; B^{0,s}\big) \quad \text{and} \quad \nabla_h^\alpha u \in \widetilde{L}^2\big(0, T; B^{0,s}\big), \qquad \partial_3 u \in \widetilde{L}^2\big(0, T; B^{0,s}\big).$$

The continuity of the solution u can be proved by the same method as [11] (Theorem 3.1). For the completeness, we give the details as follows. From (1.1), we have

$$\Delta_k^v u_t = -\nu_h \Delta_k^v D_h^\alpha u + \nu_3 \partial_3^2 u - \Delta_k^v (u \cdot \nabla u) - \Delta_k^v \nabla \Pi.$$

We can easily obtain that for all $k \ge -1$,

$$\frac{d}{dt} \big\|\Delta_k^v u(t)\big\|_{L^2}^2 = -\nu_h \big\|\nabla_h^\alpha \Delta_k^v u\big\|_{L^2}^2 - \nu_3 \big\|\partial_3 \Delta_k^v u\big\|_{L^2}^2 - \big(\Delta_k^v (u \cdot \nabla u)|\Delta_k^v u\big)_{L^2}.$$

Obviously, we have

$$\nu_h \big\|\nabla_h^\alpha \Delta_k^v u\big\|_{L^2}^2, \nu_3 \big\|\partial_3 \Delta_k^v u\big\|_{L^2}^2 \in L^1\big([0,T]\big).$$

From Lemma 1, applying the method which we have used in (3.8), one can obtain that

$$\big(\Delta_k^v(u\cdot\nabla u)|\Delta_k^v u\big)_{L^2} \in L^1\big([0,T]\big).$$

Then, we have $\frac{d}{dt}\|\Delta_k^v u(t)\|_{L^2}^2 \in L^1([0,T])$ for all $k \geq -1$. Combining with the fact that $u \in \widetilde{L}^\infty(0,T;B^{0,s})$, we easily get that $u \in C([0,T];B^{0,s})$.

3.2 Uniqueness

In this subsection, we will prove the uniqueness of the solution which obtained in the above subsection. For this end, we assume that u_1 and u_2 are the two solutions of (1.1) with the same initial data u_0, and

$$u_i \in C\big([0,T];B^{0,s}\big), \quad \nabla_h^\alpha u_i \in L^2\big(0,T;B^{0,s}\big), \quad \partial_3 u_i \in L^2\big(0,T;B^{0,s}\big), \quad i=1,2.$$

Let $w := u_1 - u_2$, we have

$$\partial_t w + \nu_h D_h^{2\alpha} w - \nu_3 \partial_3^2 w + \nabla p = -w\cdot\nabla u_2 - u_1\cdot\nabla w. \tag{3.14}$$

Applying Δ_k^v to (3.14), taking the L^2 inner product of the resulting equation with $\Delta_k^v w$, then we infer that

$$\begin{aligned}\frac{d}{dt}\big\|\Delta_k^v w\big\|_{L^2}^2 &+ 2\nu_h \big\|\nabla_h^\alpha \Delta_k^v w\big\|_{L^2}^2 + 2\nu_3 \big\|\partial_3 \Delta_k^v w\big\|_{L^2}^2 \\ &\leq 2\big|\big(\Delta_k^v(u_1\cdot\nabla w)|\Delta_k^v w\big)_{L^2}\big| + 2\big|\big(\Delta_k^v(w\cdot\nabla u_2)|\Delta_k^v w\big)_{L^2}\big|.\end{aligned} \tag{3.15}$$

The first term of right side of (3.15) can be written as

$$\big(\Delta_k^v(u_1\cdot\nabla w)|\Delta_k^v w\big)_{L^2} = \big(\Delta_k^v\big(u_1^h\cdot\nabla_h w\big)|\Delta_k^v w\big)_{L^2} + \big(\Delta_k^v\big(u_1^3\partial_3 w\big)|\Delta_k^v w\big)_{L^2}.$$

Applying the method as [3] (Lemma 1), we can easily get the bound of the right hand sides of (3.15). Indeed, for the first term of right side of (3.15), it can be written as

$$\big(\Delta_k^v(u_1\cdot\nabla w)|\Delta_k^v w\big)_{L^2} = \big(\Delta_k^v\big(u_1^h\cdot\nabla_h w\big)|\Delta_k^v w\big)_{L^2} + \big(\Delta_k^v\big(u_1^3\partial_3 w\big)|\Delta_k^v w\big)_{L^2}.$$

We give the paraproduct algorithm of J. -M. Bony(see [2]), in the vertical which reads

$$\begin{aligned}\Delta_k^v\big(u_1^h\cdot\nabla_h w\big) = &\sum_{|k-k'|\leq N_0} \Delta_k^v\big(S_{k'-1}^v u_1^h \Delta_{k'}^v w\big) + \sum_{|k-k'|\leq N_0} \Delta_k^v\big(S_{k'-1}^v w \Delta_{k'}^v u_1\big) \\ &+ \sum_{k'>k-N_0} \Delta_k^v\big(\Delta_{k'}^v u_1 \widetilde{\Delta}_{k'}^v w\big).\end{aligned}$$

In this section, we just consider the case $s \in [\frac{1}{2}, \frac{3}{2}]$. For the case $s > \frac{3}{2}$, the uniqueness was proved in [3]. Thus by the praproduct law in 2-D Sobolev space and interpolation inequality, we have

$$\begin{aligned}
&\sum_{|k-k'|\leq \leq N_0} \Delta_k^v\big(S_{k'-1}^v u_1^h \Delta_{k'}^v \nabla_h w\big)|\Delta_k^v w)_{L^2} \\
&\leq \sum_{|k-k'|\leq N_0} \big\|S_{k'-1}^v u_1\big\|_{L_v^\infty(L_h^2)} \big\|\Delta_{k'}^v \nabla_h w\big\|_{L_v^2(\dot H^{\frac{3}{4}}(\mathbb{R}^2))} \big\|\Delta_k^v w\big\|_{L_v^2(\dot H^{\frac{1}{4}}(\mathbb{R}^2))} \\
&\leq C d_k^2 2^{-2k(s-1)} \|u_1\|_{B^{0,s}}^{\frac{1}{2}} \|\partial_3 u_1\|_{B^{0,s}}^{\frac{1}{2}} \|w\|_{B^{0,s-1}}^{2-\frac{2}{\alpha}} \big\|\nabla_h^\alpha w\big\|_{B^{0,s-1}}^{\frac{2}{\alpha}} \\
&\leq C d_k^2 2^{-2k(s-1)} \bigg(C\|u_1\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}} \|\partial_3 u_1\|_{B^{0,s-1}}^{\frac{\alpha}{2\alpha-1}} \|w\|_{B^{0,s-1}}^2 + \frac{\nu_h}{10} \big\|\nabla_h^\alpha w\big\|_{B^{0,s-1}}^2 \bigg).
\end{aligned}$$

Similarly, according the same method, we also have

$$\begin{aligned}
&\sum_{|k-k'|\leq N_0} \Delta_k^v\big(S_{k'-1}^v \nabla_h w \Delta_{k'}^v u_1^h\big)|\Delta_k^v w)_{L^2} \\
&\leq \sum_{|k-k'|\leq N_0} \big\|S_{k'-1}^v \nabla_h w\big\|_{L_v^\infty(\dot H^{\frac{3}{4}}(\mathbb{R}^2))} \big\|\Delta_{k'}^v u_1\big\|_{L^2} \big\|\Delta_k^v w\big\|_{L_v^2(\dot H^{\frac{1}{4}}(\mathbb{R}^2))} \\
&\leq \sum_{\substack{|k-k'|\leq N_0 \\ k''\leq k'-2}} \big\|\Delta_{k''}^v \nabla_h w\big\|_{L_v^2(\dot H^{\frac{3}{4}}(\mathbb{R}^2))} 2^{k''(s-1)} \big\|\Delta_{k'}^v u_1\big\|_{L^2} 2^{\frac{1}{2}k'} \\
&\quad \times \big\|\Delta_k^v w\big\|_{L_v^2(\dot H^{\frac{1}{4}}(\mathbb{R}^2))} 2^{(k''-k')(\frac{3}{2}-s)} 2^{-k'(s-1)} \\
&\leq C d_k^2 2^{-2k(s-1)} \|u_1\|_{B^{0,s}}^{\frac{1}{2}} \|\partial_3 u_1\|_{B^{0,s}}^{\frac{1}{2}} \|w\|_{B^{0,s-1}}^{2-\frac{2}{\alpha}} \big\|\nabla_h^\alpha w\big\|_{B^{0,s-1}}^{\frac{2}{\alpha}} \\
&\leq C d_k^2 2^{-2k(s-1)} \bigg(C\|u_1\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}} \|\partial_3 u_1\|_{B^{0,s-1}}^{\frac{\alpha}{2\alpha-1}} \|w\|_{B^{0,s-1}}^2 + \frac{\nu_h}{10} \big\|\nabla_h^\alpha w\big\|_{B^{0,s-1}}^2 \bigg)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k'>k-N_0} \Delta_k^v\big(\Delta_{k'}^v u_1^h \Delta_{k'}^v \nabla_h w\big)|\Delta_k^v w)_{L^2} \leq \sum_{k'>k-N_0} \big\|\Delta_{k'}^v \nabla_h w\big\|_{L_v^2(\dot H^{\frac{3}{4}}(\mathbb{R}^2))} 2^{k'(s-1)} \\
&\quad \times \big\|\Delta_{k'}^v u_1\big\|_{L^2} 2^{\frac{1}{2}k'} \big\|\Delta_k^v w\big\|_{L_v^2(\dot H^{\frac{1}{4}}(\mathbb{R}^2))} 2^{-k'(s-\frac{1}{2})} 2^{\frac{1}{2}k} \\
&\leq C d_k^2 2^{-2k(s-1)} \|u_1\|_{B^{0,s}}^{\frac{1}{2}} \|\partial_3 u_1\|_{B^{0,s}}^{\frac{1}{2}} \|w\|_{B^{0,s-1}}^{2-\frac{2}{\alpha}} \big\|\nabla_h^\alpha w\big\|_{B^{0,s-1}}^{\frac{2}{\alpha}} \\
&\leq C d_k^2 2^{-2k(s-1)} \bigg(C\|u_1\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}} \|\partial_3 u_1\|_{B^{0,s-1}}^{\frac{\alpha}{2\alpha-1}} \|w\|_{B^{0,s-1}}^2 + \frac{\nu_h}{10} \big\|\nabla_h^\alpha w\big\|_{B^{0,s-1}}^2 \bigg).
\end{aligned}$$

In the last inequality, we have used the assumption that $s \geq \frac{1}{2}$. The same method used in the second term of the right hand sides of (3.15) implies that

$$\begin{aligned}
&\big(\Delta_k^v(w\cdot\nabla u_2)|\Delta_k^v w\big)_{L^2} \\
&\quad \leq C d_k^2 2^{-2k(s-1)} \big\|\nabla_h^{\frac{1}{4}} w\big\|_{B^{0,s-1}} \|\nabla u_2\|_{B^{0,s}} \big\|\nabla_h^{\frac{3}{4}} w\big\|_{B^{0,s-1}} \\
&\quad \leq d_k^2 2^{-2k(s-1)} \bigg(C\|\partial_3 u_2\|_{B^{0,s}}^{\frac{2\alpha}{2\alpha-1}} \|w\|_{B^{0,s-1}}^2 + \frac{\nu_h}{10} \big\|\nabla_h^\alpha w\big\|_{B^{0,s-1}}^2 \bigg) \\
&\qquad + C d_k^2 2^{-2k(s-1)} \|u_2\|_{B^{0,s}}^{\frac{2\alpha-2}{2\alpha-1}} \big\|\nabla_h^\alpha u_2\big\|_{B^{0,s}}^{\frac{2}{2\alpha-1}} \|w\|_{B^{0,s-1}}^2.
\end{aligned}$$

To estimate the term $(\Delta_k^v(u_1^3\partial_3 w)|\Delta_k^v w)_{L^2}$, let us again using Bony decomposition in the vertical variable:

$$\Delta_k^v\big(u_1^3\partial_3 w\big) = F_k^{v,1} + F_k^{v,2}$$

with

$$F_k^{v,1}{=}\Delta_k^v\bigg(\sum_{k'\geq k-N_0} S_{k'+2}^v(\partial_3 w)\Delta_{k'}^v u_1^3\bigg), \quad F_k^{v,2}{=}\Delta_k^v\bigg(\sum_{|k'-k|\leq N_0} S_{k'-1}^v u_1^3\partial_3\Delta_{k'}^v w\bigg).$$

Bernstein's inequality, along with Proposition 1 yields

$$\begin{aligned}\big(F_k^{v,1}|\Delta_k^v w\big)_{L^2} \leq d_k^2 2^{-2k(s-1)}\Big(&C\|\partial_3 u_1\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}}\|w\|_{B^{0,s-1}}^2 \\ &+\frac{\nu_h}{10}\big\|\nabla_h^\alpha w\big\|_{B^{0,s-1}}^2\Big)\|w\|_{B^{0,s-1}}^2,\end{aligned} \tag{3.16}$$

where $\sum_{k\geq -1} d_k \leq 1$. To get the estimate of the term $(F_k^{v,2}|\Delta_k^v w)_{L^2(\mathbb{R}^3)}$, we have to use in a crucial way the structure of the nonlinearity. Following a computation done in [4], we get that

$$\big(F_k^{v,2}|\Delta_k^v w\big)_{L^2} = \big(S_k^v u_1^3\partial_3\Delta_k^v w|\Delta_k^v w\big)_{L^2} + R_k\big(u_1^3, w\big)$$

with $R_k(u_1^3, w)$ defined by

$$\begin{aligned}&\sum_{|k-k'|\leq N_0}\big(\big[\Delta_k^v, S_{k'-2}u_1^3\big]\partial_3\Delta_{k'}^v w|\Delta_k^v w\big)_{L^2} \\ &\quad+\sum_{|k-k'|\leq N_0}\big(S_k^v - S_{k'-2}^v\big)u_1^3\partial_3\Delta_{k'}^v w|\Delta_k^v w)_{L^2}.\end{aligned}$$

Then, using an integration by parts, we infer that

$$\big(S_k^v u^3\partial_3\Delta_k^v w|\Delta_k^v w\big)_{L^2} = -\frac{1}{2}\big(S_k^v\big(\partial_3 u_1^3\big)\Delta_k^v w|\Delta_k^v w\big)_{L^2}. \tag{3.17}$$

So according the same method which discussed in Lemma1, we finally have

$$\begin{aligned}\big(\Delta_k^v\big(u_1^3\cdot\partial_3 w\big)|\Delta_k^v w\big)_{L^2} \leq Cd_k^2 2^{-2k(s-1)}\bigg(&C\|u_1\|_{B^{0,s}}^{\frac{\alpha}{2\alpha-1}}\|\partial_3 u_1\|_{B^{0,s-1}}^{\frac{\alpha}{2\alpha-1}}\|w\|_{B^{0,s-1}}^2 \\ &+\frac{\nu_h}{10}\big\|\nabla_h^\alpha w\big\|_{B^{0,s-1}}^2\bigg).\end{aligned} \tag{3.18}$$

Integrating (3.15) over $[0,T]$, combining the above estimates (3.15)–(3.18), one can obtain that

$$\begin{aligned}&\big\|w(t)\big\|_{B^{0,s-1}}^2 + \nu_h\big\|\nabla_h^\alpha w(t)\big\|_{\widetilde{L}_t^2(B^{0,s-1})}^2 + \nu_3\|\partial_3 w\|_{\widetilde{L}_t^2(B^{0,s-1})}^2 d\tau \\ &\leq C\int_0^t\big(\|\partial_3 u_1\|_{B^{0,s-1}}^{\frac{\alpha}{2\alpha-1}} + \|\partial_3 u_2\|_{B^{0,s-1}}^{\frac{2\alpha}{2\alpha-1}}\big)\|w\|_{B^{0,s-1}}^2 d\tau \\ &\quad+C\int_0^t\big\|\nabla_h^\alpha u_2\big\|_{B^{0,s}}^{\frac{2}{2\alpha-1}}\|w\|_{B^{0,s-1}}^2 d\tau\end{aligned} \tag{3.19}$$

for all $t \in [0,T]$. Then using Gronwall's lemma to (3.16), we deduce that $w \equiv 0$ on [0,T] and the uniqueness has been proved.

3.3 Blow-up criterion

In this subsection, we will prove the blow-up criterion (3.1). First, let us denote by T^* the maximal time of existence of system (1.1). Applying Δ_k^v to (1.1) and taking the L^2 inner product of the resulting equation with $\Delta_k^v u$, then performing a time integration, we can infer that

$$\begin{aligned}&\|u\|_{L_t^\infty(B^{0,s})}+\sqrt{\nu_h}\big\|\nabla_h^\alpha u\big\|_{L_t^2(B^{0,s})}+\sqrt{\nu_3}\|\partial_3 u\|_{L_t^2(B^{0,s})}\\&\le C\sum_{k\ge -1}2^{ks}\Big(\int_0^t\big|\big(\Delta_k^v(u\cdot\nabla u)|\Delta_k^v u\big)_{L^2}\big|d\tau\Big)^{\frac12}\end{aligned}$$

for all $t\in[0,T^*)$. Using Lemma 1 and interpolation inequality, we obtain

$$\begin{aligned}\big|\big(\Delta_k^v(u\cdot\nabla u)|\Delta_k^v u\big)_{L^2}\big|&\le Cd_k 2^{-2ks}\big(\|u\|^{\frac12}_{L_v^\infty(L_h^2)}\|\nabla_h u\|^{\frac12}_{L_v^\infty(L_h^2)}\|u\|^{\frac12}_{B^{0,s}}\|\nabla_h u\|^{\frac32}_{B^{0,s}}\\&\quad+\|\nabla_h u\|_{L_v^\infty(L_h^2)}\|u\|_{B^{0,s}}\|\nabla_h u\|_{B^{0,s}}\big)\\&\le Cd_k^2 2^{-2ks}\big(\|u\|^{1-\frac{1}{2\alpha}}_{L_v^\infty(L_h^2)}\big\|\nabla_h^\alpha u\big\|^{\frac{1}{2\alpha}}_{L_v^\infty(L_h^2)}\|u\|^{2-\frac{3}{2\alpha}}_{B^{0,s}}\big\|\nabla_h^\alpha u\big\|^{\frac{3}{2\alpha}}_{B^{0,s}}\\&\quad+\|u\|^{1-\frac{1}{\alpha}}_{L_v^\infty(L_h^2)}\big\|\nabla_h^\alpha u\big\|^{\frac{1}{\alpha}}_{L_v^\infty(L_h^2)}\|u\|^{2-\frac{1}{\alpha}}_{B^{0,s}}\big\|\nabla_h^\alpha u\big\|^{\frac{1}{\alpha}}_{B^{0,s}}\\&\quad+\|u\|^{1-\frac{1}{2\alpha}}_{L_v^\infty(L_h^2)}\big\|\nabla_h^\alpha u\big\|^{\frac{1}{2\alpha}}_{L_v^\infty(L_h^2)}\|u\|^{2-\frac{1}{2\alpha}}_{B^{0,s}}\big\|\nabla_h^\alpha u\big\|^{\frac{1}{2\alpha}}_{B^{0,s}}\big)\\&\le d_k^2 2^{-2ks}\frac{\nu_h}{2}\big\|\nabla_h^\alpha u\big\|^2_{B^{0,s}}\\&\quad+Cd_k^2 2^{-2ks}\nu_h^{-\frac{3}{4\alpha-3}}\|u\|^{\frac{4\alpha-2}{4\alpha-3}}_{L_v^\infty(L_h^2)}\big\|\nabla_h^\alpha u\big\|^{\frac{2}{4\alpha-3}}_{L_v^\infty(L_h^2)}\|u\|^2_{B^{0,s}}\\&\quad+Cd_k^2 2^{-2ks}\nu_h^{-\frac{1}{2\alpha-1}}\|u\|^{\frac{2\alpha-2}{2\alpha-1}}_{L_v^\infty(L_h^2)}\big\|\nabla_h^\alpha u\big\|^{\frac{2}{2\alpha-1}}_{L_v^\infty(L_h^2)}\|u\|^2_{B^{0,s}}\\&\quad+Cd_k^2 2^{-2ks}\nu_h^{-\frac{1}{4\alpha-1}}\|u\|^{\frac{4\alpha-2}{4\alpha-1}}_{L_v^\infty(L_h^2)}\big\|\nabla_h^\alpha u\big\|^{\frac{2}{4\alpha-1}}_{L_v^\infty(L_h^2)}\|u\|^2_{B^{0,s}}.\end{aligned}$$

The Gronwall's Lemma gives

$$\begin{aligned}\big\|u(t)\big\|_{B^{0,s}}\le\|u_0\|_{B^{0,s}}\exp\Big\{\int_0^t&\big(\nu_h^{-\frac{3}{4\alpha-3}}\big\|u(\tau)\big\|^{\frac{4\alpha-2}{4\alpha-3}}_{L_v^\infty(L_h^2)}\big\|\nabla_h^\alpha u\big\|^{\frac{2}{4\alpha-3}}_{L_v^\infty(L_h^2)}\\&+\nu_h^{-\frac{1}{4\alpha-1}}\big\|u(\tau)\big\|^{\frac{4\alpha-2}{4\alpha-1}}_{L_v^\infty(L_h^2)}\big\|\nabla_h^\alpha u\big\|^{\frac{2}{4\alpha-1}}_{L_v^\infty(L_h^2)}\\&+\nu_h^{-\frac{1}{2\alpha-1}}\|u\|^{\frac{2\alpha-2}{2\alpha-1}}_{L_v^\infty(L_h^2)}\big\|\nabla_h^\alpha u\big\|^{\frac{2}{2\alpha-1}}_{L_v^\infty(L_h^2)}\big)d\tau\Big\}.\end{aligned}\tag{3.20}$$

Then if the right hand sides of (3.20) are bounded, then $T^*=\infty$. Otherwise, if T^* is finite, we can define the value of $u(T^*)$ by $\lim_{t\to T^*}u(t)$ in $B^{0,s}$. Using $u(T^*)$ as a new initial data, applying the local existence result which have obtained before, we extend u beyond T^*. This ends the proof of Theorem 2.

4 Global Solution of System (1.1)

In order to show that the solution constructed in the previous section can be continued to the global one, we need the following *a priori* estimate. Note that in the present work, we assume that $u_0 \in B^{0,s}$, for $s \geq 0$. However, it seems not enough to get the higher regularities of u. Fortunately, the local existence result which has been obtained in previous section supplies a unique solution u satisfies

$$u \in C\big([0,T];B^{0,s}\big) \quad \text{and} \quad \nabla_h^\alpha u \in \widetilde{L}^2\big(0,T;B^{0,s}\big), \qquad \partial_3 u \in \widetilde{L}^2\big(0,T;B^{0,s}\big)$$

for some positive T, which implies $\nabla_h^\alpha u, \partial_3 u \in L^2(0,T;H^{0,s})$. Thus, there exists a $0 < T_0 < T$ such that $\nabla_h^\alpha u(T_0), \partial_3 u(T_0) \in L^2(\mathbb{R}^3)$. Without loss of generality, we assume $\nabla_h^\alpha u_0, \partial_3 u_0 \in L^2(\mathbb{R}^3)$. Using this fact, we have the following lemmas.

Lemma 2. *Under the assumptions of Theorem* 1*, there exists a universal constant C such that for all $t \in [0,T^*)$, the following* a priori *estimate holds true:*

$$\begin{aligned}&\big\|\nabla_h^\alpha u\big\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2 \\ &\quad + \int_0^t \|\partial_t u\|_{L^2}^2 + \big\|D_h^{2\alpha} u\big\|_{L^2}^2 + \big\|\partial_3^2 u\big\|_{L^2}^2 + \big\|\partial_3 \nabla_h^\alpha u\big\|_{L^2}^2 d\tau \\ &\quad \leq C\big(\big\|\nabla_h^\alpha u_0\big\|_{L^2}^2 + \|\partial_3 u_0\|_{L^2}^2 + t\big) \exp\Big\{\int_0^t \big\|\nabla_h^\alpha u\big\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2 d\tau\Big\},\end{aligned} \tag{4.1}$$

where T^* is maximal lifespan of (1.1)*.*

Proof. First, taking the L^2 scalar product with $\partial_t u$ in (1.1), we obtain that

$$\|\partial_t u\|_{L^2}^2 + \frac{1}{2}\nu_h \frac{d}{dt}\big\|D_h^\alpha u\big\|_{L^2}^2 + \frac{1}{2}\nu_3 \frac{d}{dt}\|\partial_3 u\|_{L^2}^2 \leq \frac{1}{2}\|\partial_t u\|_{L^2}^2 + C\|u\cdot\nabla u\|_{L^2}^2,$$

that is

$$\big\|\partial_t u^h\big\|_{L^2}^2 + \nu_h \frac{d}{dt}\big\|D_h^\alpha u\big\|_{L^2}^2 + \nu_3 \frac{d}{dt}\|\partial_3 u\|_{L^2}^2 \leq C\|u\cdot\nabla u\|_{L^2}^2. \tag{4.2}$$

In order to estimate the convection term, we split $\|u\cdot\nabla u\|_{L^2}$ into

$$\|u\cdot\nabla u\|_{L^2} \leq \big\|u^h\cdot\nabla_h u\big\|_{L^2} + \big\|u^3\cdot\partial_3 u\big\|_{L^2} := I + II.$$

For I, by Hölder inequality, Sobolev embedding theorem and estimate (1.2), one gets

$$\begin{aligned} I &\leq \big\|u^h\big\|_{L_h^\infty(L_v^4)} \|\nabla_h u\|_{L_h^2(L_v^4)} \\ &\leq C\big\|u^h\big\|_{L_h^\infty(L_v^2)}^{\frac{3}{4}} \big\|\partial_3 u^h\big\|_{L_h^\infty(L_v^2)}^{\frac{1}{4}} \|\nabla_h u\|_{L_h^2(L_v^2)}^{\frac{3}{4}} \|\partial_3\nabla_h u\|_{L_h^2(L_v^2)}^{\frac{1}{4}} \\ &\leq C\big\|u^h\big\|_{L^2}^{\frac{3}{2}-\frac{3}{2\alpha}} \big\|\nabla_h^\alpha u\big\|_{L^2}^{\frac{3}{2\alpha}} \|\partial_3 u\|_{L^2}^{\frac{1}{2}-\frac{1}{2\alpha}} \big\|\partial_3\nabla_h^\alpha u\big\|_{L^2}^{\frac{1}{2\alpha}}. \end{aligned} \tag{4.3}$$

For II, by Hölder inequality and Sobolev embedding theorem, one gets

$$\begin{aligned} II &\leq \big\|u^3\partial_3 u\big\|_{L^2(\mathbb{R}^3)} \\ &\leq C\big\|u^3\big\|_{L_h^{\frac{8}{3}}(L_v^\infty)} \|\partial_3 u\|_{L_h^8(L_v^2)} \\ &\leq C\big\|u^3\big\|_{L_h^{\frac{8}{3}}(L_v^2)}^{\frac{1}{2}} \big\|\partial_3 u^3\big\|_{L_h^{\frac{8}{3}}(L_v^2)}^{\frac{1}{2}} \|\partial_3 u\|_{L_h^8(L_v^2)} \\ &\leq C\big\|\nabla_h^{\frac{1}{4}} u^3\big\|_{L^2}^{\frac{1}{2}} \big\|\nabla_h^{\frac{1}{4}}\partial_3 u^3\big\|_{L^2}^{\frac{1}{2}} \big\|\partial_3\nabla_h^{\frac{3}{4}} u\big\|_{L^2}. \end{aligned} \tag{4.4}$$

Using the divergence free condition $\operatorname{div} u = 0$ again, one gets

$$\big\|\nabla_h^{\frac14}\partial_3 u^3\big\|_{L^2} = \big\|\nabla_h^{\frac14}\operatorname{div}_h u^h\big\|_{L^2} \le C\|u\|_{L^2}^{1-\frac{5}{4\alpha}}\big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{5}{4\alpha}}.$$

This fact together with (4.4) and basic energy estimate (1.2) yields

$$II \le C\big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{3}{4\alpha}}\|\partial_3 u\|_{L^2}^{1-\frac{3}{4\alpha}}\big\|\partial_3\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{3}{4\alpha}}.$$

Next, we consider the following Stokes system

$$\begin{cases}\nu_h D_h^{2\alpha}u - \nu_3\partial_3^2 u + \nabla \Pi = -\partial_t u - u\cdot\nabla u,\\ \operatorname{div} u = 0.\end{cases}$$

Taking L^2 estimate of elliptic equation yields

$$\begin{aligned}&\nu_h\big\|D_h^{2\alpha}u\big\|_{L^2}^2 + \nu_3\big\|\partial_3^2 u\big\|_{L^2}^2 + (\nu_h+\nu_3)\big\|\partial_3\nabla_h^{\alpha}u\big\|_{L^2}^2 + \|\nabla\Pi\|_{L^2}^2\\ &\quad\le C_1\big(\|\partial_t u\|_{L^2}^2 + \|u\cdot\nabla u\|_{L^2}^2\big).\end{aligned}\tag{4.5}$$

Making use of (4.3)–(4.4) in (4.5), we have

$$\begin{aligned}&\nu_h\big\|D_h^{2\alpha}u\big\|_{L^2}^2 + \nu_3\big\|\partial_3^2 u\big\|_{L^2}^2 + (\nu_h+\nu_3)\big\|\partial_3\nabla_h^{\alpha}u\big\|_{L^2}^2 + \|\nabla\Pi\|_{L^2}^2\\ &\quad\le C_1\Big(\|\partial_t u\|_{L^2}^2 + \big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{6}{2\alpha-1}}\|\partial_3 u\|_{L^2}^{\frac{2\alpha-2}{2\alpha-1}} + \|\partial_3 u\|_{L^2}^2\big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{6}{4\alpha-3}}\Big).\end{aligned}\tag{4.6}$$

Hence, combining (4.6) with (4.2), applying Young's inequality implies

$$\begin{aligned}&\frac{d}{dt}\Big(\big\|D_h^{\alpha}u\big\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2\Big) + \|\partial_t u\|_{L^2}^2 + \big\|D_h^{2\alpha}u\big\|_{L^2}^2 + \big\|\partial_3^2 u\big\|_{L^2}^2 + \big\|\partial_3\nabla_h^{\alpha}u\big\|_{L^2}^2\\ &\quad\le C\big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{6}{2\alpha-1}}\|\partial_3 u\|_{L^2}^{\frac{2\alpha-2}{2\alpha-1}} + C\|\partial_3 u\|_{L^2}^2\big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{6}{4\alpha-3}}.\end{aligned}$$

Performing a time integration on $[0,t]$ yields

$$\begin{aligned}&\big\|\nabla_h^{\alpha}u\big\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2 + \int_0^t\Big(\|\partial_t u\|_{L^2}^2 + \big\|D_h^{2\alpha}u\big\|_{L^2}^2 + \big\|\partial_3^2 u\big\|_{L^2}^2 + \big\|\partial_3\nabla_h^{\alpha}u\big\|_{L^2}^2\Big)d\tau\\ &\quad\le C\big\|\nabla_h^{\alpha}u_0\big\|_{L^2}^2 + C\|\partial_3 u_0\|_{L^2}^2 + C\int_0^t\big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{4}{\alpha}}d\tau\\ &\qquad + C\int_0^t\Big(\big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{6}{2\alpha-1}}\|\partial_3 u\|_{L^2}^{\frac{2\alpha-2}{2\alpha-1}} + \|\partial_3 u\|_{L^2}^2\big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{6}{4\alpha-3}}\Big)d\tau.\end{aligned}$$

Since $\alpha \ge \frac32$, according to Young's inequality, one can deduce that

$$\begin{aligned}\int_0^t\big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{6}{2\alpha-1}}\|\partial_3 u\|_{L^2}^{\frac{2\alpha-2}{2\alpha-1}}d\tau &\le C\int_0^t\Big(\big\|\nabla_h^{\alpha}u\big\|_{L^2}^4 + \|\partial_3 u\|_{L^2}^4 + 1^{\frac{8\alpha-4}{6\alpha-8}}\Big)d\tau,\\ \int_0^t\|\partial_3 u\|_{L^2}^2\big\|\nabla_h^{\alpha}u\big\|_{L^2}^{\frac{6}{4\alpha-3}}d\tau &\le C\int_0^t\Big(\big\|\nabla_h^{\alpha}u\big\|_{L^2}^4 + \|\partial_3 u\|_{L^2}^4 + 1^{\frac{2\alpha-\frac32}{2\alpha-3}}\Big)d\tau.\end{aligned}$$

Finally, we obtain that

$$\begin{aligned}&\big\|\nabla_h^{\alpha}u\big\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2 + \int_0^t\Big(\|\partial_t u\|_{L^2}^2 + \big\|D_h^{2\alpha}u\big\|_{L^2}^2 + \big\|\partial_3^2 u\big\|_{L^2}^2 + \big\|\partial_3\nabla_h^{\alpha}u\big\|_{L^2}^2\Big)d\tau\\ &\quad\le C\Big(\big\|\nabla_h^{\alpha}u_0\big\|_{L^2}^2 + \|\partial_3 u_0\|_{L^2}^2 + t\Big) + C\int_0^t\Big(\big\|\nabla_h^{\alpha}u\big\|_{L^2}^4 + \|\partial_3 u\|^4\Big)d\tau.\end{aligned}$$

Then Gronwall's inequality completes the proof of this lemma. □

Remark 1. Combining Lemma 2 with (1.2), we find that

$$\int_0^t \left\|\nabla_h^\alpha u\right\|_{L^2}^2 + \|\partial_3 u\|_{L^2}^2 d\tau \leq C$$

for all $t \in [0, T^*)$. This means the right hand side of (4.1) is finite.

Completion of the proof of the Theorem 1.1. First, we treat on the case that $\nabla_h^\alpha u_0 \in L^2(\mathbb{R}^3)$ and $\partial_3 u_0 \in L^2(\mathbb{R}^3)$. For $\alpha \geq \frac{3}{2}$, Lemma 2 implies that $\nabla_h^{2\alpha} u$, $\partial_3^2 u$, $\partial_3 \nabla_h^\alpha u$ belong to $L^2(0, T^*; L^2)$. If $T^* < +\infty$, by Hölder inequality, we have

$$\begin{aligned}
&\int_0^t \left\|u(\tau)\right\|_{L_v^\infty(L_h^2)}^{\frac{4\alpha-2}{4\alpha-3}} \left\|\nabla_h^\alpha u(\tau)\right\|_{L_v^\infty(L_h^2)}^{\frac{2}{4\alpha-3}} d\tau \\
&\leq C \int_0^t \|u\|_{L^2}^{\frac{2\alpha-1}{4\alpha-3}} \|\partial_3 u\|_{L^2}^{\frac{2\alpha-1}{4\alpha-3}} \left\|\partial_3 \nabla_h^\alpha u\right\|_{L^2}^{\frac{1}{4\alpha-3}} \left\|\nabla_h^\alpha u\right\|_{L^2}^{\frac{1}{4\alpha-3}} d\tau \\
&\leq C \|u\|_{L_t^\infty(L^2)}^{\frac{2\alpha-1}{4\alpha-3}} \|\partial_3 u\|_{L_t^\infty(L^2)}^{\frac{2\alpha-1}{4\alpha-3}} \left\|\nabla_h^\alpha u\right\|_{L_t^\infty(L^2)}^{\frac{1}{4\alpha-3}} \left\|\partial_3 \nabla_h^\alpha u\right\|_{L_t^2(L^2)}^{\frac{1}{4\alpha-3}} \leq C
\end{aligned}$$

for all $t \in [0, T^*)$. Similarly,

$$\int_0^t \left\|u(\tau)\right\|_{L_v^\infty(L_h^2)}^{\frac{4\alpha-2}{4\alpha-1}} \left\|\nabla_h^\alpha u(\tau)\right\|_{L_v^\infty(L_h^2)}^{\frac{2}{4\alpha-1}} d\tau \leq C$$

and

$$\int_0^t \left\|u(\tau)\right\|_{L_v^\infty(L_h^2)}^{\frac{2\alpha-2}{2\alpha-1}} \left\|\nabla_h^\alpha u(\tau)\right\|_{L_v^\infty(L_h^2)}^{\frac{2}{2\alpha-1}} d\tau \leq C$$

for all $t \in [0, T^*)$. Hence the blow up criterion satisfied, which implies $T^* = +\infty$.

Next, when $u_0 \in B^{0,s}(\mathbb{R}^3)$, Theorem 2 supplies a unique solution u with

$$u \in C\big([0, T^*), B^{0,s}\big) \quad \text{and} \quad \nabla_h^\alpha u \in \widetilde{L}^2\big(0, T^*; B^{0,s}\big), \quad \partial_3 u \in \widetilde{L}^2\big(0, T^*; B^{0,s}\big).$$

Hence there exists a $T_0 < T^*$ such that $\nabla_h^\alpha u(T_0)$, $\partial_3 u(T_0)$ belong to $L^2(\mathbb{R}^3)$. In the previous case we get a unique solution $\widetilde{u}$ on $[T_0, \infty)$, with data $u(T_0)$. Uniqueness ensures that $\widetilde{u} \equiv u$ on $[T_0, T^*)$. Therefore u can be continued globally.

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